PRIFYSGOL ABERYSTWYTH UNIVERSITY

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 2 / SEMESTER 2 EXAMINATIONS

MAI / MAY 2020

MA21410 - Linear Algebra

The questions on this paper are written in English.

If you have questions about the paper during the exam, contact the module co-ordinator, Dr Rob Douglas, on rsd@aber.ac.uk.

You should write out solutions to the paper and upload them to Blackboard as a single PDF file.

Amser a ganiateir - 3 awr
Mae'n rhaid cyflwyno eich atebion erbyn 12:30 (amser y DU).

- Gellir rhoi cynnig ar bob cwestiwn.
- Rhoddir mwy o ystyriaeth i berfformiad yn rhan B wrth bennu marc dosbarth cyntaf.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

Time allowed - 3 hours
Submission must be completed by 12:30 (UK time).

- All questions may be attempted.
- Performance in section B will be given greater consideration in assigning a first class mark.
- Students may submit answers to this paper in either Welsh or English.

Section A

1. Determine which of the following subsets of $\mathbb{R}^3$ are subspaces.

(a) $S_1 = \{(x, y, z) : x + y + z = 13\}$; [3 marks]

(b) $S_2 = \{(x, y, z) : 2x - 4y + 7z = 0\}$; [5 marks]

(c) $S_3 = \{(x, y, z) : x - 6y + 2z \geq 0\}$; [4 marks]

(d) $S_4 = \{(x, y, z) : x^2 + y^6 + z^4 = 0\}$. [4 marks]

2. Determine which of the following subsets of $\mathbb{R}^3$ are linearly independent.

(a) $\{(1, -5, 3), (0, 0, 7), (0, 0, 0)\}$; [2 marks]

(b) $\{(7, 3, 4), (1, 1, 0), (7, 3, 4)\}$; [2 marks]

(c) $\{(1, 2, 4), (2, 1, 3), (4, -1, 1)\}$; [4 marks]

(d) $\{(1, 2, 1), (1, 1, 0), (6, 2, 3)\}$. [4 marks]

3. Let $\mathcal{P}_3(\mathbb{R})$ be the vector space of polynomials (in the variable t) of degree less than or equal to 3 with real coefficients. Define the subspace $\mathcal{F} \subset \mathcal{P}_3(\mathbb{R})$ by

$$\mathcal{F} = \{(3a + b)t^3 + (2b - 5c)t^2 + (a - b + c)t - 6a : a, b, c \in \mathbb{R}\}.$$

Find a spanning set for $\mathcal{F}$. [3 marks]

4. Let $M_{2\times 2}(\mathbb{R})$ be the vector space of 2×2 matrices with real entries. Define the subspace $\mathcal{M} \subset M_{2\times 2}(\mathbb{R})$ by

$$\mathcal{M} = \text{sp}\left(\begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 5 \\ 2 & -2 \end{pmatrix}, \begin{pmatrix} 5 & 5 \\ 8 & 0 \end{pmatrix}\right).$$

Prove that dim$\mathcal{M} = 2$, where dim$\mathcal{M}$ denotes the dimension of $\mathcal{M}$. [5 marks]

5. Let $T : \mathcal{P}_3(\mathbb{R}) \to \mathbb{R}^4$ be defined by

$$T(at^3 + bt^2 + ct + d) = (a + b, b + c, c + d, a + d),$$

where $a, b, c, d \in \mathbb{R}$.

(a) Prove that T is a linear transformation. [4 marks]

(b) Obtain the matrix representing T with respect to standard bases. [4 marks]

(c) Find the kernel of T and the image space of T, and give bases for both. [4,5 marks]

(d) Verify the conclusions of the dimension theorem for the linear transformation T. [2 marks]

6. Find all the eigenvalues and eigenspaces of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(x) = Ax$, where

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 4 & 0 \\ -3 & 6 & 2 \end{pmatrix}.$$

[9 marks]

Giving appropriate justification, show that A can be diagonalised. Write down an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix. [6 marks]

Section B

7. Define subspaces $W_1, W_2 \subset \mathbb{R}^2$ by $W_1 = \text{sp}((2,0))$ and $W_2 = \text{sp}((1,1))$. Demonstrate that $W_1 \cup W_2$ is not a subspace of $\mathbb{R}^2$. [4 marks]

8. Let $\mathcal{P}_2(\mathbb{R})$ denote the vector space of polynomials (in the variable x) with degree less than or equal to 2 with real coefficients. Show that $\{x^2+2x+6, x^2+4x+4, x^2\}$ spans $P_2(\mathbb{R})$, but that $\{x^2+2x, 3x+8\}$ does not. [5,2 marks]

9. Let V be a real vector space, and let $u, v, w \in V$.

(a) Explain what it means for $\{u, v, w\}$ to be a *basis* for V. [4 marks]

(b) Given that $\{u, v, w\}$ is a basis for V, prove that $\{u+v, u+3v, u+v+2w\}$ is a basis for V. [6 marks]

10. Let $M_{2\times 2}(\mathbb{R})$ denote the vector space of 2×2 matrices with real entries. Define

$$\mathcal{M} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : b + c + d = 0 \right\},$$

for $a, b, c, d \in \mathbb{R}$. Demonstrate that $\mathcal{M}$ is a subspace of $M_{2\times 2}(\mathbb{R})$; determine its dimension. [4,3 marks]

11. Let $M_{3\times 3}(\mathbb{R})$ be the vector space of 3×3 matrices with real entries. Suppose $\mathcal{M}_1$, $\mathcal{M}_2$ are subspaces of $M_{3\times 3}(\mathbb{R})$, with $\dim\mathcal{M}_1 = 4$, $\dim\mathcal{M}_2 = 7$, where $\dim\mathcal{M}$ denotes the dimension of a subspace $\mathcal{M}$. Further suppose $\mathcal{M}_1$ is not contained in $\mathcal{M}_2$, and $\mathcal{M}_1 + \mathcal{M}_2 \neq M_{3\times 3}(\mathbb{R})$. Determine $\dim(\mathcal{M}_1 \cap \mathcal{M}_2)$. [5 marks]

12. Let V be a vector space, and suppose $V = \text{sp}(v_1, v_2, v_3)$ for some $v_1, v_2, v_3 \in V$. Further suppose a linear transformation $T : V \to V$ is surjective. Prove that $\{T(v_1), T(v_2), T(v_3)\}$ spans V. [4 marks]

13. Let V be a vector space, and let $S : V \to V$ be a linear transformation.

(a) Explain what is meant by the *kernel of* S, denoted $\text{Ker}(S)$, and the *image space* of S, denoted $\text{Im}(S)$. [2 marks]

(b) Suppose that S is *idempotent*, that is $S^2 = S$, where $S^2(v) = S(S(v))$ for $v \in V$. Prove that $V = \text{Im}(S) \oplus \text{Ker}(S)$, where $\oplus$ denotes the direct sum of subspaces. [5 marks]

14. Let $A \in M_{n\times n}(\mathbb{C})$, the vector space of $n \times n$ matrices with complex entries. Suppose A is *unitary*, that is $A^*A = AA^* = I$, where A^* denotes the Hermitian (conjugate) transpose of A, and I the identity matrix. Prove that all eigenvalues of A have modulus 1. [6 marks]