PRIFYSGOL ABERYSTWYTH UNIVERSITY

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 2 / SEMESTER 2 EXAMINATIONS

MAI / MAY 2020

MA21510 - Complex Analysis

The questions on this paper are written in English.

If you have questions about the paper during the exam, contact the module co-ordinator, Dr Adil Mughal, on aqm@aber.ac.uk.

You should write out solutions to the paper and upload them to blackboard as a single PDF file.

Amser a ganiateir - 3 awr
Mae'n rhaid cyflwyno eich atebion erbyn 12:30 (amser y DU).

- Gellir rhoi cynnig ar bob cwestiwn.
- Rhoddir mwy o ystyriaeth i berfformiad yn rhan B wrth bennu marc dosbarth cyntaf.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

Time allowed - 3 hours
Submission must be completed by 12:30 (UK time).

- All questions may be attempted.
- Performance in section B will be given greater consideration in assigning a first class mark.
- Students may submit answers to this paper in either Welsh or English.

Section A

1. Sketch the portion of the complex plane corresponding to the following inequalities.

(a) $Re(z+1) \geq 0$,

(b) $|z-1+i| \leq 2$,

(c) $|z-1| \leq |z+1|$. [3+3+4=10 marks]

2. Find all the complex numbers, in the form $z = x + iy$ with $x, y \in \mathbb{R}$, which satisfy $z^4 = -16$. [8 marks]

3. (a) State Euler's formula for the complex exponential function exp.

(b) Express $\cos(i)$ in the form $x + iy$ with x, y real.

(c) Draw the image under exp of the following subsets of the complex plane:

(i) $\{z \in \mathbb{C} : Re(z) = \frac{\pi}{4}\}$,

(ii) $\{z \in \mathbb{C} : Im(z) = \frac{\pi}{4}\}$,

(iii) $\{z \in \mathbb{C} : Im(z) > 0\}$.

[2+2+6=10 marks]

4. (a) Find the real and imaginary parts of the function $f(z) = (z+1)^2$.

(b) State the Cauchy-Riemann equations and verify them for the real and imaginary parts of $f(z) = (z+1)^2$.

(c) What can be concluded about f by verifying the Cauchy-Riemann equations?

[3+4+1=8 marks]

5. (a) State the definition of a harmonic function.

(b) Verify that $u(x,y) = x^3 + 3x^2y - 3xy^2 - y^3$ is harmonic.

(c) Find another harmonic function $v(x,y)$ such that $f(x+iy) = u(x,y) + iv(x,y)$ is analytic. [2+4+6=12 marks]

6. (a) State Cauchy's integral formula.

(b) If C is the circular contour $|z| = 2$, use Cauchy's integral formula to evaluate

$$\oint \frac{e^{\pi z}}{(z+i)} dz.$$

[3+7=10 marks]

7. (a) State the definition of the principle branch Log of the logarithm and $z^{\frac{1}{4}}$, the fourth root.

(b) Evaluate the principal values $Log(1+i)$ and $(-1+i\sqrt{3})^{\frac{1}{4}}$ (in the form $x + iy$ with x, y real).

(c) Draw the image of $\mathbb{C} \setminus \{0\}$ under Log and under $z^{\frac{1}{4}}$. [5+5+2=12 marks]

Section B

8. Evaluate the following integrals,

(a)

$$\int_{C_1} |z|^2 dz$$

where C_1 is the straight line segment from -1 to 3i,

(b)

$$\int_{C_2} \sin z dz$$

where C_2 is given by the parametrisation $\gamma : [0, \frac{\pi}{2}] \ni t \mapsto \frac{1}{2}t + i\cos(5t)$.

[7+8=15 marks]

9. Compute explicitly the Laurent series around $z = 0$ of the function

$$f(z) = \frac{1}{z^2(z+5)}.$$

[8 marks]

10. Let z_0 be a pole of order 2 for a function f that can be written by a Laurent series around z_0. Prove that the residue can be obtained by the following formula.

$$\text{res}_f(z_0) = \lim_{z\to z_0} \frac{d}{dz}\left[(z-z_0)^2 f(z)\right]$$

All relevant facts which relate poles and their residues to the Laurent series can be treated as well known but they should be stated explicitly in your proof. [10 marks]

11. Use the Residue Theorem to evaluate the following integrals,

(a)

$$\oint_{|z-2|=1} \frac{\log(z)}{z^3 - 4z^2 + 4z} dz,$$

(b)

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)} dx,$$

(c)

$$\int_0^{2\pi} \frac{1}{2i + \cos(x)} dx.$$

[6+5+6=17 marks]