PRIFYSGOL ABERYSTWYTH UNIVERSITY

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 2 / SEMESTER 2 EXAMINATIONS

MAI / MAY 2020

MA25220 - Introduction to Numerical Analysis and its Applications

The questions on this paper are written in English.

If you have questions about the paper during the exam, contact the module co-ordinator, Dr Tudur Davies, on itd@aber.ac.uk.

You should write out solutions to the paper and upload them to blackboard as a single PDF file.

Amser a ganiateir - 3 awr
Mae'n rhaid cyflwyno eich atebion erbyn 12:30 (amser y DU).

- Gellir rhoi cynnig ar bob cwestiwn.
- Rhoddir mwy o ystyriaeth i berfformiad yn rhan B wrth bennu marc dosbarth cyntaf.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

Time allowed - 3 hours
Submission must be completed by 12:30 (UK time).

- All questions may be attempted.
- Performance in section B will be given greater consideration in assigning a first class mark.
- Students may submit answers to this paper in either Welsh or English.

Section A

1. For the following data:

i	0	1	2
x_i	-1	0	1
$f(x_i)$	1	3	0

(a) Construct the piecewise linear Lagrange interpolant. [5 marks]

(b) Construct the quadratic Lagrange interpolating polynomial. [5 marks]

2. (a) Construct the forward difference table for the following data:

x	0.0	0.1	0.2	0.3	0.4	0.5
$f(x)$	1.00	0.62	0.28	-0.02	-0.28	-0.5

[6 marks]

(b) What is the lowest degree of polynomial which matches the data exactly? [1 mark]

(c) Determine the interpolating polynomial using the forward difference formula

$$p_n(x) = \sum_{i=0}^{n} \binom{r}{i} \Delta^i f(x_0)$$

where $r = (x - x_0)/h$. [5 marks]

3. Recall that the error formula for the composite Trapezoidal rule for approximating

$$I(f) = \int_a^b f(x)dx$$

using n subintervals with width $h = (b - a)/n$ is

$$E_{T_n} = -\frac{(b-a)}{12} h^2 f''(\xi)$$

for some $\xi \in [a, b]$.

Determine the number of subintervals required so that the composite Trapezoidal rule estimates the integral

$$I(f) = \int_0^1 \frac{dx}{1+x}$$

correct to six decimal places.

(*You are not asked to compute a numerical estimate of the integral.*) [8 marks]

4. Simpson's rule for approximating the integral $\int_a^b f(x)dx$ is given by

$$S_2 = \frac{h}{3}\Big[f(x_0) + 4f(x_1) + f(x_2)\Big],$$

where $x_0 = a$, $x_1 = (a+b)/2$, $x_2 = b$ and $h = (b-a)/2$.

(a) By dividing the interval $[a, b]$ into $2N$ subintervals of equal length, derive the composite Simpson's rule. [5 marks]

(b) A function $f(x)$ is given by the table of values below.

x	0	1	2	3	4
$f(x)$	2	7	12	10	5

Use Simpson's rule to estimate the integral $\int_0^4 f(x)dx$ with

(i) $n = 2$ subintervals;

(ii) $n = 4$ subintervals.

[6 marks]

5. (a) Show that the function

$$g(x) = \sqrt[4]{x+10}$$

is a contraction mapping on $[0, 6]$. [6 marks]

(b) Use the function $g(x)$ from part (a) and a suitable starting point to compute the root of the equation

$$x^4 - x - 10 = 0$$

correct to three decimal places. [4 marks]

6. (a) Show that the equation

$$2x^2 + 5 = e^x$$

has a root in the interval $[3, 4]$. [3 marks]

(b) Use Newton-Raphson iteration to find the root accurate to five decimal places. [6 marks]

7. (a) Show that the initial value problem

$$y' = y - 1, \quad y(0) = 1/4,$$

has a unique solution for $0 \le x \le 1$. [4 marks]

(b) Solve the initial value problem given in part (a) using Euler's method with step size $h = 1/4$. [6 marks]

Section B

8. Construct the natural cubic spline $s(x)$ that interpolates the following data:

i	0	1	2
x_i	0	2	4
$f(x_i)$	3	0	2

[14 marks]

9. (a) Find the values of the weights ω_0, ω_1 and ω_2 so that the quadrature rule

$$\int_{-1}^{1} f(x)dx \approx \omega_0 f\left(-\sqrt{\frac{3}{5}}\right) + \omega_1 f(0) + \omega_2 f\left(\sqrt{\frac{3}{5}}\right)$$

is exact for polynomials of (at least) degree 2. [8 marks]

(b) Use this rule to estimate

$$I(f) = \int_0^2 te^{2t}dt.$$

[4 marks]

10. (a) Obtain a quadratic equation associated with the iterative method

$$x_{n+1} = \frac{bx_n^2 + 2cx_n}{c - ax_n^2}.$$

[3 marks]

(b) Show that if $c \neq 0$ and $ax^2 \neq c$ at a root x^* then the method is exactly second order. [7 marks]

11. Consider the initial value problem

$$y' = -y + x + 1, \;\; 0 \leq x \leq 1, \;\; y(0) = 1.$$

(a) Use the Simple Runge-Kutta method

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n))\right]$$

to solve this IVP with $h = 0.5$. [4 marks]

(b) Show that this Runge-Kutta method is consitent with the IVP. [3 marks]

(c) What restriction on h, if any, is required for this Runge-Kutta method to be absolutely stable? [7 marks]