PRIFYSGOL ABERYSTWYTH UNIVERSITY

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 2 / SEMESTER 2 EXAMINATIONS

MAI / MAY 2020

MA25610 - Hydrodynamics I

The questions on this paper are written in English.

If you have questions about the paper during the exam, contact the module co-ordinator, Simon Cox, on sxc@aber.ac.uk.

You should write out solutions to the paper and upload them to blackboard as a single PDF file.

Amser a ganiateir - 3 awr
Maen rhaid cyflwyno eich atebion erbyn 12:30yb (amser y DU).

Time allowed - 3 hours
Submission must be completed by 12:30pm (UK time).

- Gellir rhoi cynnig ar bob cwestiwn.
- Rhoddir mwy o ystyriaeth i berfformiad yn rhan B wrth bennu marc dosbarth cyntaf.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

- All questions may be attempted.
- Performance in section B will be given greater consideration in assigning a first class mark.
- Students may submit answers to this paper in either Welsh or English.

Useful formulae

In the usual notation:

$$\underline{\nabla}\phi = \frac{1}{h_1}\frac{\partial\phi}{\partial x_1}\underline{e}_1 + \frac{1}{h_2}\frac{\partial\phi}{\partial x_2}\underline{e}_2 + \frac{1}{h_3}\frac{\partial\phi}{\partial x_3}\underline{e}_3$$

$$\underline{\nabla}.\underline{u} = \frac{1}{h_1h_2h_3}\left(\frac{\partial(h_2h_3u_1)}{\partial x_1} + \frac{\partial(h_1h_3u_2)}{\partial x_2} + \frac{\partial(h_1h_2u_3)}{\partial x_3}\right)$$

$$\underline{\nabla} \wedge \underline{u} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\underline{e}_1 & h_2\underline{e}_2 & h_3\underline{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ h_1u_1 & h_2u_2 & h_3u_3 \end{vmatrix}$$

$$\underline{\nabla}^2\phi = \frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial x_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\phi}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial x_3}\right)\right)$$

Section A

1. Determine the pathlines for the following velocity fields given in Cartesian and plane polar coordinates, where λ is a positive constant. In each case sketch the pathlines on Cartesian axes.

(a) $\underline{u} = \lambda x \underline{e}_x - \lambda y \underline{e}_y$, [6 marks]

(b) $\underline{u} = \lambda y \underline{e}_x$, [6 marks]

(c) $\underline{u} = \lambda r \underline{e}_\theta$. [6 marks]

2. (a) Give an expression for the convective derivative of a vector field $\underline{u}$. [3 marks]

(b) Calculate the convective derivative of the velocity field

$$\underline{u} = 3xt\underline{e}_x - 3yzt\underline{e}_y + z\underline{e}_z.$$

[6 marks]

3. (a) The vorticity of a fluid flow is given by $\underline{\omega} = \underline{\nabla} \wedge \underline{u}$. Calculate the vorticity of the following fluid flows:

(i) $\underline{u} = x\underline{e}_x - y\underline{e}_y$, [3 marks]

(ii) $\underline{u} = x\underline{e}_y$, [3 marks]

(iii) $\underline{u} = \dfrac{a}{r}\underline{e}_r$, [3 marks]

(iv) $\underline{u} = ar\underline{e}_\theta$, [3 marks]

where a is a positive constant.

(b) State which, if any, of these flows are irrotational. [2 marks]

4. Determine whether or not the following flows are suitable for incompressible fluids:

(a) $\underline{u} = x\underline{e}_x - y\underline{e}_y$, [3 marks]

(b) $\underline{u} = \dfrac{a}{r}\underline{e}_r$, [3 marks]

where a is a positive constant.

5. Does a stream function ψ exist for each of the following velocity fields? Either state a reason why not or give a suitable stream function.

(a) $\underline{u} = \Omega y \underline{e}_x - \Omega x \underline{e}_y$ [3 marks]

(b) $\underline{u} = \Omega x \underline{e}_x + \Omega y \underline{e}_y$ [3 marks]

(c) $\underline{u} = \dfrac{\Omega}{r^2} \underline{e}_\theta$, [3 marks]

where Ω is a positive constant.

6. A two-dimensional fluid flow has the following velocity potential:

$$\phi(x, y) = 3\mathrm{e}^{5x} \sin(5y).$$

(a) Determine the velocity field. [4 marks]

(b) Show that ϕ is harmonic. [3 marks]

7. Penelope Penguin glides gracefully through the sea off Antarctica at an average speed of three metres per second. Derive an estimate for the Reynolds number of the flow of water past her body, and hence explain whether or not an inviscid fluid would provide a good model for this flow. [7 marks]

Section B

8. Consider the two-dimensional velocity field $\underline{u} = t\underline{e}_x + \frac{3}{y}\underline{e}_y$, where t is time.

(a) Give an equation for the streamline through the point (x_0, y_0). [4 marks]

(b) Show that the pathline through the point (x_0, y_0) has equation

$$72(x - x_0) = \left(y^2 - y_0^2\right)^2 .$$

[6 marks]

9. Water flows from a tap of diameter 1cm, through a hosepipe of diameter 2cm, and then out of a short nozzle with diameter 5mm. The water can be assumed incompressible.

(a) If liquid leaves the tap at 1 m/s, at what speed does it flow out of the nozzle? [6 marks]

(b) Taking the acceleration due to gravity to be $g = 10\mathrm{m/s^2}$, if the hosepipe descends by hm over its length, what is the change in water pressure between the tap and the nozzle in terms of h and the water density ρ? [7 marks]

10. (a) The velocity potential for a particular fluid flow is found by combining the potential for a uniform stream with fluid speed U, $\phi_1 = Ux$, with the potential for a doublet of strength μ, $\phi_2 = \frac{\mu \cos\theta}{r}$. Find a streamfunction for this flow. [8 marks]

(b) Give an expression for the stream function governing the flow with velocity field

$$\underline{u} = -U\cos\theta\left(1 - \frac{R^2}{r^2}\right)\underline{e}_r + U\sin\theta\left(1 + \frac{R^2}{r^2}\right)\underline{e}_\theta,$$

where R is a constant. [5 marks]

11. An homogeneous inviscid liquid extends to infinity where the pressure is zero. In it is a spherical bubble of gas in which the pressure (assumed to be constant throughout the bubble) is inversely proportional to the bubble volume to the power of $4/3$. The fluid is initially at rest, with the radius of the bubble equal to a and the pressure in the bubble equal to p_0.

Let $R(t)$ be the radius of the bubble. At a given radial position r in the fluid, R satisfies

$$\frac{R^2\ddot{R}}{r} + \frac{2R\dot{R}^2}{r} - \frac{R^4\dot{R}^2}{2r^4} = \frac{p}{\rho},$$

where p and ρ are the pressure and density of the liquid respectively.

(a) Show that the pressure p in the bubble satisfies

$$p = p_0\left(\frac{a}{R}\right)^4.$$

[4 marks]

(b) Use the identity

$$2R^2\left(R\ddot{R} + \frac{3}{2}\dot{R}^2\right) = \frac{\mathrm{d}}{\mathrm{d}R}\left(R^3\dot{R}^2\right),$$

to show that in the subsequent motion R satisfies the equation

$$\rho\dot{R}^2 = 2p_0\left(\frac{a^3}{R^3} - \frac{a^4}{R^4}\right).$$

[10 marks]