PRIFYSGOL ABERYSTWYTH UNIVERSITY

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 2 / SEMESTER 2 EXAMINATIONS

MAI / MAY 2020

MA25710 - Advanced Dynamics

The questions on this paper are written in English.

If you have questions about the paper during the exam, contact the module co-ordinator, Dr Adil Mughal, on aqm@aber.ac.uk.

You should write out solutions to the paper and upload them to blackboard as a single PDF file.

Amser a ganiateir - 3 awr
Mae'n rhaid cyflwyno eich atebion erbyn 12:30 (amser y DU).

- Gellir rhoi cynnig ar bob cwestiwn.
- Rhoddir mwy o ystyriaeth i berfformiad yn rhan B wrth bennu marc dosbarth cyntaf.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

Time allowed - 3 hours
Submission must be completed by 12:30 (UK time).

- All questions may be attempted.
- Performance in section B will be given greater consideration in assigning a first class mark.
- Students may submit answers to this paper in either Welsh or English.

Section A

1. (a) The **radius** of the event horizon of a black hole depends on the **gravitational constant** G, the **mass** of the black hole M and the **speed** of light c.

(i) Using dimensional analysis find an expression for the radius of the event horizon. Note the gravitational constant has dimensions of $[\text{length}]^3[\text{mass}]^{-1}[\text{time}]^{-2}$. [8 marks]

(ii) What happens to the radius of the event horizon if the mass of the black hole is doubled? [2 marks]

(b) The drag **force** on an object (i.e. air resistance) depends on the **density** ρ of the air, the **velocity** V of the object and its cross-sectional **area** A.

(i) Using dimensional analysis find an expression for the drag force on the object. [8 marks]

(ii) What happens to the drag force if the velocity of the object is doubled? [2 marks]

2. Find the extremal $y(x)$ for the following functional,

$$I = \int_{x_1}^{x_2} 2y(x)\sin x - (y'(x))^2 dx,$$

where $y'(x) = \dfrac{dy(x)}{dx}$. [10 marks]

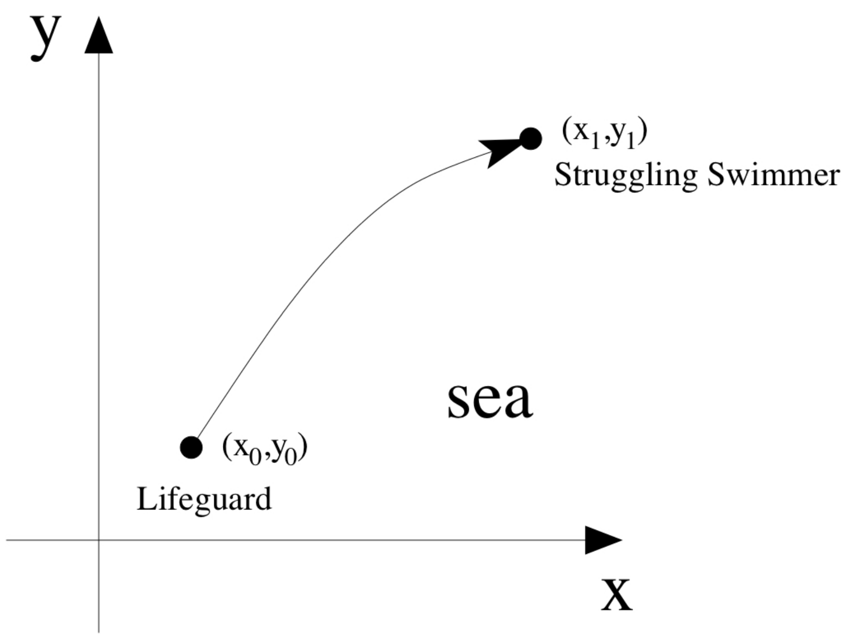
3. (a) Find the extremal $y(x)$ for the following functional,

$$I = \int_{x_1}^{x_2} (y'(x))^2 + y(x)\ dx,$$

where $y'(x) = \dfrac{dy(x)}{dx}$.

(b) Hence find the function $y(x)$ that satisfies the boundary conditions $y(0) = 0$ and $y(1) = 2$. [10 marks]

4. A lifeguard starts off at the point $(x = x_0, y = y_0)$ and has to travel in the minimum possible time to reach a struggling swimmer at the point $(x = x_1, y = y_1)$, as shown in the diagram below.



The speed at which the lifeguard travels is $v = \dfrac{v_0}{y}$, where v_0 is a constant.

(a) Write down an expression for the total time taken for the life guard to reach the swimmer and thus show that the path satisfies the equation

$$\frac{y(x)}{(1 + y'(x)^2)^{\frac{1}{2}}} = a,$$

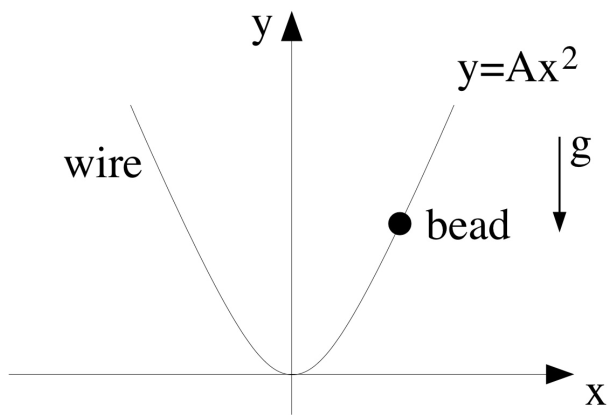
where a is a constant and $y'(x) = \dfrac{dy(x)}{dx}$. [7 marks]

(b) By direct substitution or otherwise, demonstrate that a solution of this equation is given by

$$y = a \cosh\left(\frac{x - b}{a}\right).$$

[3 marks]

5. A bead of mass m slides without friction along a wire which has the shape of a parabola $y = Ax^2$ with axis vertical in the earth's gravitational field g.



(a) Show that the Lagrangian for the bead is given by

$$\mathcal{L} = \frac{1}{2}m(1 + 4A^2x^2)\dot{x}^2 - mgAx^2.$$

where $\dot{x} = \dfrac{dx}{dt}$. [5 marks]

(b) Hence find the equation of motion for the bead. [5 marks]

6. A particle of mass m moves in a central potential $V(r) = -\beta r^k$, where β and k are constants and r is the distance of the mass from the origin.

(a) Using polar coordinates write down the Lagrangian for the system. [1 mark]

(b) (i) Show that the angular momentum for the particle is given by,

$$L = mr^2\dot{\theta}.$$

[2 marks]

(ii) Hence, show that the equation of motion for the particle is given by

$$m\ddot{r} = F_{eff}(r),$$

where $\ddot{r} = \dfrac{d^2r}{dt^2}$, and

$$F_{eff}(r) = \frac{L^2}{mr^3} - \frac{dV(r)}{dr}$$

is the effective force. [2 marks]

(c) Show that the radius of a circular orbit in this potential is given by,

$$r_0 = \left(\frac{L^2}{m\beta k}\right)^{1/(k+2)}.$$

Give the period of the orbit in terms of r_0. [5 marks]

Section B

7. Consider the functional

$$I = \int_{x_1}^{x_2} \sqrt{y(x)\left[1 + (y'(x))^2\right]}\, dx,$$

(a) Show that the first integral of the Euler-Lagrange equation for this functional is

$$y' = \frac{1}{k}\sqrt{y - k^2},$$

where k is a constant. [6 marks]

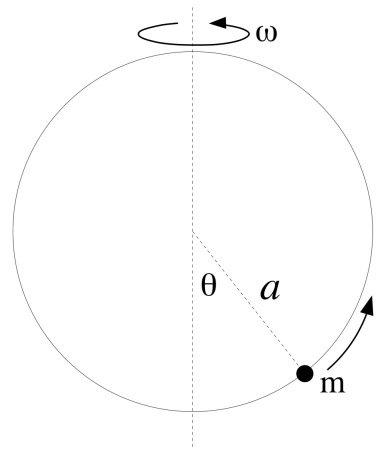
(b) By solving the resulting differential equation in part (a), show that for the boundary conditions $y(-1) = y(1) = A$ the functional is stationary on the path

$$y(x) = \frac{x^2}{4k^2} + k^2 \quad \text{with} \quad A = k^2 + \frac{1}{4k^2}.$$

[8 marks]

(c) By making the substitution $h = k^2$, determine the number of stationary paths: if $A > 1$ and if $A = 1$, clearly stating your reasons. [6 marks]

8. A particle of mass m slides under gravity and without friction on a circular hoop of radius a. The hoop is rotating with a constant angular velocity ω about a vertical axis that is a diameter of the hoop (as shown in the following diagram).



(a) In terms of the angle θ between the radius vector for the particle and the downward vertical, write down the Lagrangian for the system. [5 marks]

(b) By applying Lagrange's equation show that the equation of motion for the system is given by:

$$\ddot{\theta} = \sin\theta\left(\omega^2\cos\theta - \frac{g}{a}\right).$$

[5 marks]

(c) Suppose θ_0 is an equilibrium position and let $\theta = \theta_0 + \alpha$ (where α is a small quantity). Using the result from part (b), show that to leading order in α the equation of motion can be written as

$$\ddot{\alpha} = -\alpha\left(\frac{g}{a}\cos\theta - \omega^2\cos 2\theta_0\right).$$

Hint: You may find it useful to use the approximations:

$$\begin{aligned}\sin(\theta_0+\alpha) &\approx \sin\theta_0 + \alpha\cos\theta_0,\\ \cos(\theta_0+\alpha) &\approx \cos\theta_0 - \alpha\sin\theta_0.\end{aligned}$$

[13 marks]

(d) If $\omega^2 < \frac{g}{a}$ the point at $\theta_0 = 0$ is a position of equilibrium for the particle. Find the oscillation frequency of small amplitude vibrations about this point. [4 marks]

(e) If $\omega^2 > \frac{g}{a}$ show that the point at

$$\theta_0 = \cos^{-1}\left(\frac{g}{a\omega^2}\right)$$

instead becomes a stable equilibrium for the particle. [3 marks]