PRIFYSGOL ABERYSTWYTH UNIVERSITY

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 2 / SEMESTER 2 EXAMINATIONS

MAI / MAY 2020

MA32410 – Graphs and Networks

The questions on this paper are written in English.

If you have questions about the paper during the exam, contact the module co-ordinator, Dr Gwion Evans, on dfe@aber.ac.uk.

You should write out solutions to the paper and upload them to blackboard as a single PDF file.

Amser a ganiateir - 3 awr
Mae'n rhaid cyflwyno eich atebion erbyn 12:30 (amser y DU).

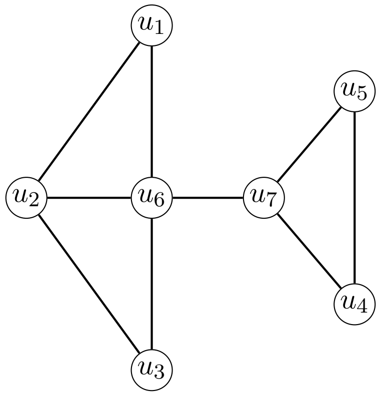
- Gellir rhoi cynnig ar bob cwestiwn.
- Rhoddir mwy o ystyriaeth i berfformiad yn rhan B wrth bennu marc dosbarth cyntaf.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

Time allowed - 3 hours
Submission must be completed by 12:30 (UK time).

- All questions may be attempted.
- Performance in section B will be given greater consideration in assigning a first class mark.
- Students may submit answers to this paper in either Welsh or English.

Section A

1. Consider the graph G whose diagram is



Give an example of each of the following in G:

(a) a non-trivial closed walk of length 6;

(b) an open path of length 6;

(c) a longest trail;

(d) a spanning tree.

[8 marks]

2. For each of the following cases, determine whether or not there exists a graph G, with vertex set V and edge list E, satisfying the stated properties. Justify your answers either by providing an example of a graph satisfying the stated properties, or by giving a brief explanation of why such a graph cannot exist.

(a) $|E| = 8$ and G is regular of degree 3.

(b) $|V| = 6$ and G has an open path of length 6.

(c) $|V| = 3$, $|E| = 4$ and G has no multiple edges.

(d) $|V| = 5$, $|E| = 4$ and G is a tree with exactly 2 bridges.

[8 marks]

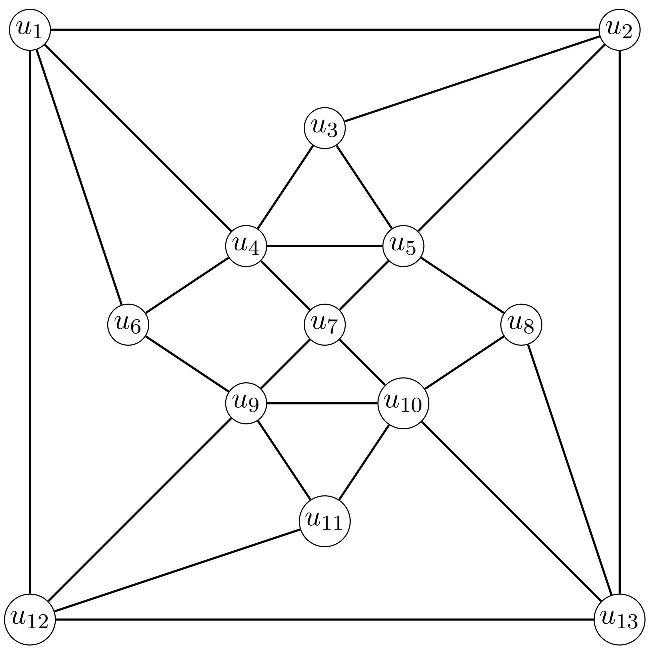
3. The matrix below is the adjacency matrix of a graph $G = (V, E)$.

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

(a) Apply the Fusion Algorithm to find the vertex set of each component of G. [6 marks]

(b) Draw a diagram of G. [2 marks]

4. Consider the graph G whose diagram is



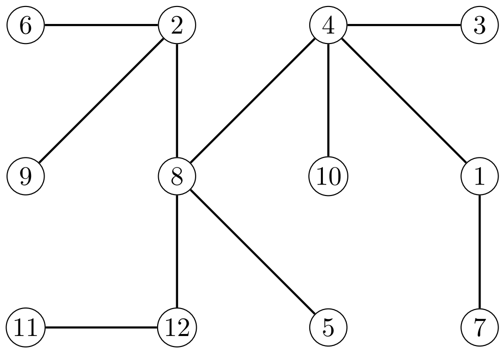
(a) Is G Eulerian?

(b) Is G Hamiltonian?

Justify your answers briefly.

[5 marks]

5. Determine the Prüfer code of the following labelled tree:



[4 marks]

6. Apply Prim's Algorithm or Kruskal's Algorithm to determine the edges and weight of a minimum weight spanning tree in the weighted graph whose weight matrix is

$$\begin{pmatrix} - & - & 3 & - & 1 & - & 2 & - & - \\ - & - & 2 & 4 & 2 & - & - & 1 & 3 \\ 3 & 2 & - & 4 & 3 & 4 & - & 2 & 4 \\ - & 4 & 4 & - & 2 & 3 & 2 & 1 & 3 \\ 1 & 2 & 3 & 2 & - & 2 & 4 & - & - \\ - & - & 4 & 3 & 2 & - & 1 & 3 & 3 \\ 2 & - & - & 2 & 4 & 1 & - & 4 & - \\ - & 1 & 2 & 1 & - & 3 & 4 & - & 3 \\ - & 3 & 4 & 3 & - & 3 & - & 3 & - \end{pmatrix}.$$

Draw a diagram of the minimum weight spanning tree you have determined. [9 marks]

7. (a) Apply the Warshall-Floyd Algorithm to the weighted digraph with weight matrix

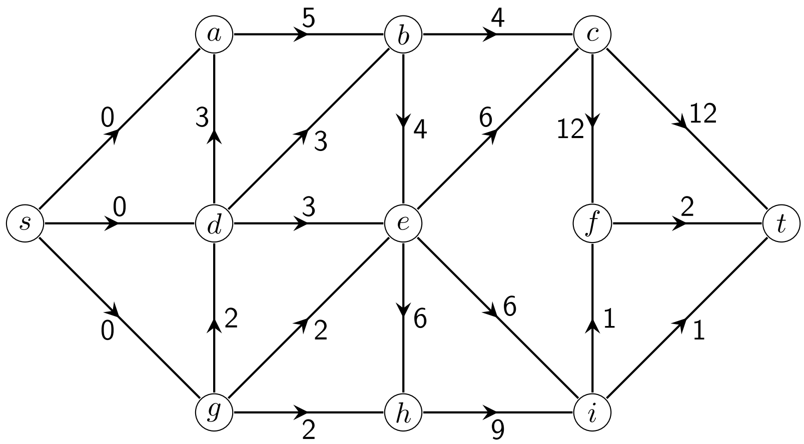
$$\begin{pmatrix} 0 & 1 & 7 & 3 & \infty \\ 4 & 0 & 9 & \infty & 2 \\ 1 & \infty & 0 & \infty & \infty \\ 2 & \infty & 2 & 0 & \infty \\ 1 & 3 & 8 & 6 & 0 \end{pmatrix}.$$

[5 marks]

(b) State the final optimum policy matrix. [3 marks]

(c) Use the final optimum policy matrix to find the shortest path from vertex 2 to vertex 3. State its length. [3 marks]

8. Below is the activity network of a project with activities $a, b, c, d, e, f, g, h, i$.



Apply the Longest Path Algorithm to find:

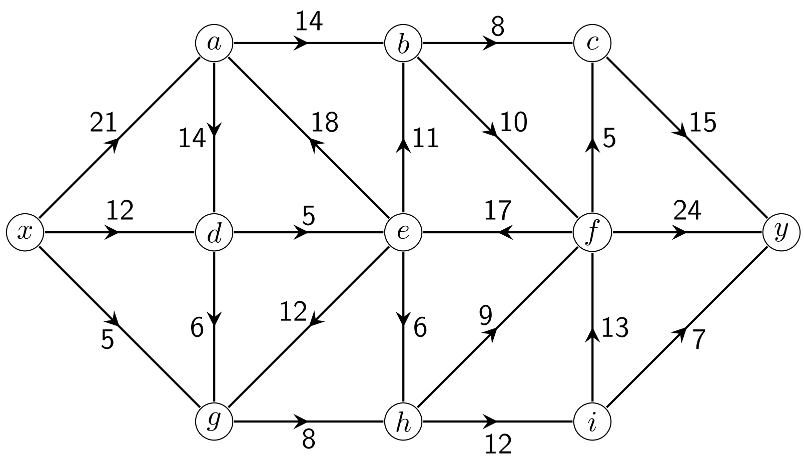
(a) the earliest start time (EST) for each activity;

(b) the earliest completion time (ECT); and

(c) a critical path.

[8 marks]

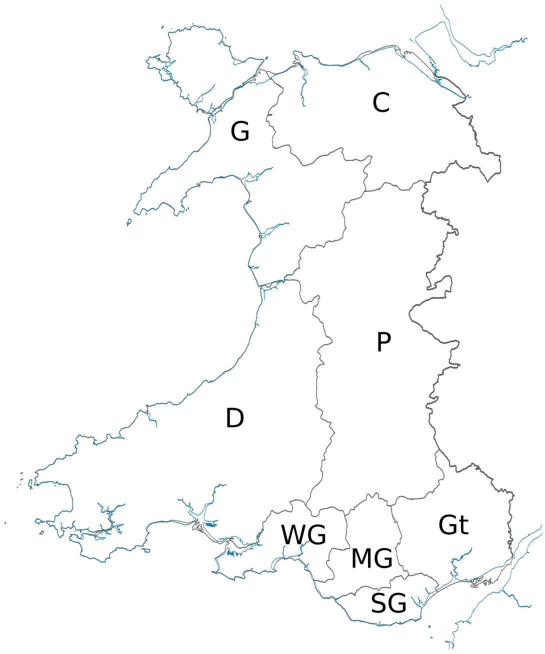
9. Apply the Ford-Fulkerson Algorithm to the following transport network. Show the final label on each vertex, the final flow along each edge and the minimum cut. State the maximum flow value.



[9 marks]

Section B

10. Consider the following map of the preserved counties of Wales, namely Gwynedd (G), Clwyd (C), Dyfed (D), Powys (P), West Glamorgan (WG), Mid Glamorgan (MG), South Glamorgan (SG) and Gwent (Gt).



(a) Imagine travel restrictions are put in place in Wales so that during the enforcement period, residents are allowed to leave their home county once only. Furthermore, residents may enter into a county from another county across any common land border, but once they leave the county they cannot return, unless it is to their home county. Travel between counties is restricted across land borders only (Gwynedd and Dyfed do not share a common land border). Travel is constrained to be within Wales only.

Is it possible for some individuals to start from their home county, travel to every other county and return to their home county during the enforcement period?

(b) Imagine that further restrictions are enforced so that travel between counties is only possible through border control points; one for each common land border between the Welsh counties. An inspector, with a valid permit that allows them to enter and leave a county as many times as is required by their work, needs to inspect each border control point. The inspector cannot pass through a border control point more than once.

(i) Starting from inside a county, is it possible for the inspector to pass through every border control point?

(ii) Is it possible for the inspector to do such a journey and finish inside the same county as the one they started from?

Justify your answers with reference to an appropriate graph, whose vertices and edges must be defined clearly and whose diagram must be drawn. [12 marks]

11. Let G be a simple, connected weighted digraph. For each edge e in G, let $W(e)$ be the weight of e and assume that $W(e) \geq 0$. Suppose that G has exactly n vertices and fix a vertex u to be the origin.

(a) During the course of applying Dijkstra's Algorithm to G a vertex x is labelled $(P(x), D(x))$. Explain what is meant by $P(x)$ and $D(x)$. [2 marks]

(b) Give a precise description of the vertex set and edge list of the shortest path tree produced by Dijkstra's Algorithm. [4 marks]

(c) For each $i = 1, \ldots, n$, let T_i be the tree formed when permanently labelling the ith vertex, according to Dijkstra's Algorithm. Fix $k \in \{2, \ldots, n\}$. Suppose that P is a path in G starting at u and ending at x, where x is the kth vertex to be permanently labelled. Give a detailed argument as to why there must exist vertices p and q such that pq is an edge of P, p is a vertex in T_{k-1} and q is not a vertex in T_{k-1}. [3 marks]

(d) Let p and q be as in part (c). Consider the values of $D(q)$ and $D(p)$ after p is permanently labelled. Explain why $D(q) \leq D(p) + W(pq)$. [3 marks]

(e) Let q and x be as in part (c). At the point when x is permanently labelled we must have that $D(x) \leq D(q)$. Explain why this must be so. [3 marks]

12. For each $k = 1, 2, 3, 4$, let $G_k = (V_k, E_k)$ be the graph with adjacency matrix A_k, as given by:

$$A_1 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad A_2 := \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix},$$

$$A_3 := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \qquad A_4 := \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

where $V_k = \{v_{1k}, v_{2k}, v_{3k}, v_{4k}\}$ and the matrices are indexed so that vertex v_{ik} corresponds to the ith row and ith column in matrix A_k.

(a) Identify which pair of the graphs are isomorphic and state a graph isomorphism from one to the other. [4 marks]

(b) For each $k = 1, 2, 3, 4$, consider each graph G_k as a labelled graph, where the labelling map $\ell_k : V_k \to \{1, 2, 3, 4\}$ is given by $\ell_k(v_{ik}) = i$. Are the isomorphic pair of graphs that you have identified in part (a) isomorphic as labelled graphs? Justify your answer. [4 marks]

13. (a) Identify the leaves of the labelled tree with Prüfer code 1,2,2,5,7,5,3 without constructing the labelled tree. Justify your answer. [3 marks]

(b) Suppose that G is a labelled tree with 100 vertices and that the the Prüfer code of G begins with 3, 3, 4, 2, 8, 10, 7, 7. Is it possible that the vertex labelled 100 is the only leaf adjacent to the vertex labelled 3? Justify your answer. [3 marks]

(c) Let T be any tree with k edges. Let G be a simple graph in which every vertex has degree at least k. Prove by induction on k that G has a subgraph isomorphic to T. [6 marks]

(d) Use part (c) to prove that a simple graph in which all vertices have degree at least k contains an open path of length k. [3 marks]