PRIFYSGOL ABERYSTWYTH UNIVERSITY

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 2 / SEMESTER 2 EXAMINATIONS

MAI / MAY 2020

MA34210 - Asymptotic Methods in Mechanics

The questions on this paper are written in English.

If you have questions about the paper during the exam, contact the module co-ordinator, Dr Adam Vellender, on asv2@aber.ac.uk.

You should write out solutions to the paper and upload them to Blackboard as a single PDF file.

Amser a ganiateir - 3 awr
Mae'n rhaid cyflwyno eich atebion erbyn 12:30 (amser y DU).

- Gellir rhoi cynnig ar bob cwestiwn.
- Rhoddir mwy o ystyriaeth i berfformiad yn rhan B wrth bennu marc dosbarth cyntaf.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

Time allowed - 3 hours
Submission must be completed by 12:30 (UK time).

- All questions may be attempted.
- Performance in section B will be given greater consideration in assigning a first class mark.
- Students may submit answers to this paper in either Welsh or English.

Formulae:
You may find the following Taylor-Maclaurin series and definite integrals useful:

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}, \quad \text{for all } x; \qquad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \text{for all } x;$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad \text{for } |x| < 1; \qquad \int_{-\infty}^{\infty} e^{-x^2} \mathrm{d}x = \sqrt{\pi}.$$

Section A

1. Let

$$f(x) = x^3 \cos\left(\frac{3}{x}\right), \qquad g(x) = 6x^3, \qquad h(x) = x^2.$$

(a) Prove that $f(x) = O(g(x))$ as $x \to \infty$. [4 marks]

(b) Prove that $g(x) = o(h(x))$ as $x \to 0$. [4 marks]

(c) Write down a function a (not equal to f) for which $f(x) \sim a(x)$ as $x \to 3/(2\pi)$. [2 marks]

(d) Is the asymptotic formula $g(x) = O(f(x))$ as $x \to 0$ valid? Justify your answer. [5 marks]

2. Obtain asymptotic approximations of the form $x = x_0 + \varepsilon x_1 + O(\varepsilon^2),\ \varepsilon \to 0$, for all three roots of the regularly perturbed cubic equation

$$x^3 - 2x - 2\varepsilon = 0,$$

where $\varepsilon > 0$ is a small parameter. [10 marks]

3. Consider the quintic equation

$$1 - 2x + \frac{\sqrt{3}}{2}\varepsilon^2 x^2 - \varepsilon^8 x^5 = 0,$$

where $\varepsilon > 0$ is a small parameter.

(a) How many roots in $\mathbb{C}$ does the quintic equation have? [1 mark]

(b) Write down the limit problem (the problem corresponding to $\varepsilon = 0$). How many roots does it have? [2 marks]

(c) Hence state (briefly explaining your reasoning) whether the quintic equation is regularly or singularly perturbed. [2 marks]

(d) Draw the Kruskal-Newton graph for this problem. [5 marks]

(e) In what form(s) would you seek to approximate the roots for this problem? State also the number of roots that each ansatz would be able to approximate.

Note: *You are not required to approximate the roots here, but only to give a suitable form or forms.* [5 marks]

4. Consider the following function $\mathcal{I} : (0,1) \to \mathbb{R}$, which is defined by an integral:

$$\mathcal{I}(x) = \int_0^x e^{-t^2} \cos(t)\mathrm{d}t,$$

(a) Obtain an asymptotic expansion for $\mathcal{I}(x)$ as $x \to 0$, giving your answer in the form $\mathcal{I}(x) = c_0 x + c_1 x^3 + c_2 x^5 + O(x^7)$, $x \to 0$, where c_0, c_1, c_2 are constants you should determine. [9 marks]

(b) Hence, or otherwise, state whether each of the following asymptotic statements is true or false (you are not required to justify your assertions):

(i) $\mathcal{I}(x) = O(x), \quad x \to 0;$
(ii) $\mathcal{I}(x) = o(x), \quad x \to 0;$
(iii) $\mathcal{I}(x) \sim x, \quad x \to 0.$ [3 marks]

5. (a) Explain briefly what secular terms are and why they are often undesirable in approximations of physical phenomena. [3 marks]

(b) Give an example of a physical system whose analysis through straightforward asymptotics may result in undesired secular terms. [1 mark]

(c) Name an asymptotic method that involves the elimination of secular terms. [1 mark]

(d) Identify all secular terms in the equation [2 marks]

$$x(t) = 3\cos t + 5\sin(3t) - 3t\sin t + 4t^2\cos^2 t + e^{-t}t\cos(2t).$$

6. Let $\eta \gg 1$ be a large positive parameter. Consider the ordinary differential equation

$$3\eta\psi''(x) = (5 - x^3)\psi(x),$$

with boundary conditions $\psi(0) = 3$ and $\psi(2) = 8$.

(a) State the order of the ODE, and whether it is linear or non-linear. [2 marks]

(b) Find a two-term asymptotic approximation of the solution ψ in the form

$$\psi(x) = \psi_0(x) + \eta^{-1}\psi_1(x) + O(\eta^{-2}), \quad \eta \to \infty.$$

[9 marks]

Section B

7. Consider the integral

$$\mathcal{K}(\lambda) = \int_{-1}^{1} f(t)e^{-\lambda g(t)}\mathrm{d}t$$

where the functions f and g are given by

$$f(t) = \sin t + 2, \qquad g(t) = 5 - 3\cos t.$$

(a) For which value of t does the function $g(t)$ take its minimum value over $(-1, 1)$? What is the value of $g(t)$ at that point? [2 marks]

(b) Using Laplace's method, perform a sequence of approximations to demonstrate that

$$\mathcal{K}(\lambda) \sim \alpha\lambda^{\beta}e^{\gamma\lambda}, \quad \lambda \to \infty,$$

where α, β, γ are constants you should determine. Give brief explanations of the approximations used throughout your working. [10 marks]

8. Consider the singularly perturbed cubic equation

$$\varepsilon^2 x^3 + 2\varepsilon x^2 + 3x - 1 = 0,$$

where $\varepsilon > 0$ is a small parameter. This equation has one regularly perturbed root and two singularly perturbed roots in $\mathbb{C}$.

(a) Explain what is meant by stating that a problem involving a small parameter is *singularly perturbed*. [2 marks]

(b) State and solve the limit problem corresponding to $\varepsilon = 0$ to obtain a leading-order (one term) approximation of the regularly perturbed root. [2 marks]

(c) Use the singular perturbation ansatz

$$x(\varepsilon) = \varepsilon^{-1}b(\varepsilon), \quad b(0) \neq 0,$$

to find approximations for both of the singularly perturbed solutions in the form

$$x = c_{\{-1\}}\varepsilon^{-1} + c_{\{0\}}\varepsilon^{0} + O(\varepsilon), \quad \varepsilon \to 0,$$

where $c_{\{-1\}}$ and $c_{\{0\}}$ are constants to be determined for each singularly perturbed root.

[10 marks]

9. Use the Lindstedt-Poincaré method to find a two-term asymptotic approximation of a periodic solution of the Duffing equation

$$\ddot{x} + 9x - 4\varepsilon x^3 = 0,$$

where the system is released from rest with an initial amplitude $\frac{1}{2}$ and ε is a small dimensionless positive parameter. [15 marks]

Hint: *You may find the following trigonometric identity useful:*

$$\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta,$$

and that solutions to the differential equation $x''(t) + x(t) = A\cos(t) + B\cos(3t)$, $A, B \in \mathbb{R}$ *are of the form*

$$x(t) = \frac{A}{2}t\sin t - \frac{1}{8}B\cos(3t) + c_1\cos t + c_2\sin t.$$

10. (a) What does it mean for a sequence of functions $\{\delta_n(x)\}$ to be an asymptotic sequence as $x \to \infty$? [2 marks]

(b) Let $a_n = (n+1)^2$ for $n \in \mathbb{N}$. Prove that the sequence of functions defined as

$$\{\delta_n(x)\} = \{e^{3x}x^{-a_n}\}$$

for $n \in \mathbb{N}$ is an asymptotic sequence as $x \to \infty$. [7 marks]