PRIFYSGOL ABERYSTWYTH UNIVERSITY

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 2 / SEMESTER 2 EXAMINATIONS

MAI / MAY 2020

MA34710 - Numerical Solution of Partial Differential Equations

The questions on this paper are written in English.

If you have questions about the paper during the exam, contact the module co-ordinator, Dr Daniel Peck, on dtp@aber.ac.uk.

You should write out solutions to the paper and upload them to blackboard as a single PDF file.

Amser a ganiateir - 3 awr
Mae'n rhaid cyflwyno eich atebion erbyn 17:00 (amser y DU).

- Gellir rhoi cynnig ar bob cwestiwn.
- Rhoddir mwy o ystyriaeth i berfformiad yn rhan B wrth bennu marc dosbarth cyntaf.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

Time allowed - 3 hours
Submission must be completed by 17:00 (UK time).

- All questions may be attempted.
- Performance in section B will be given greater consideration in assigning a first class mark.
- Students may submit answers to this paper in either Welsh or English.

Section A

1. The following finite difference schemes are designed to solve for u_m^{n+1}. For each of (i)-(v), state whether they are (a) explicit or implicit and (b) one-step or multi-step.

(i)

$$u_m^{n+1} = \alpha u_m^n - (1+\alpha)u_m^{n-1}, \quad \alpha \in \mathbb{R}$$

[3 marks]

(ii)

$$u_m^{n+1} = \alpha\left(u_{m-1}^{n+1} + u_{m-1}^n\right) - \beta u_m^n, \quad \alpha, \beta \in \mathbb{R}$$

[3 marks]

(iii)

$$u_m^{n+1} = (1+2\alpha)u_{m+1}^n - 2\alpha u_{m-1}^n, \quad \alpha \in \mathbb{R}$$

[3 marks]

(iv)

$$u_m^{n+1} = \frac{\alpha}{2}\left(u_{m+1}^{n+1} - u_{m-1}^n\right) - \frac{\beta}{2}\left(u_{m+1}^{n-1} - u_{m-1}^{n-2}\right), \quad \alpha, \beta \in \mathbb{R}$$

[3 marks]

(v)

$$u_m^{n+1} = (1+\alpha)u_{m+1}^{n-1} + \beta\left(u_{m-1}^{n+1} - u_{m-1}^n\right), \quad \alpha, \beta \in \mathbb{R}$$

[3 marks]

2. The partial differential equation defined by the linear differential operator

$$\mathcal{P}(\partial_x, \partial_t) = \partial_x^2 + a\partial_t,$$

is approximated using the following finite difference operator

$$\mathcal{P}_{k,h}\, u_m^n = \frac{1}{h^2}\left[u_{m+2}^n - 2u_{m+1}^n + u_m^n\right] + \frac{a}{k}\left[u_m^{n+1} - u_m^n\right].$$

Making use of the supplementary material given at the end of the paper, determine whether the finite difference scheme is *consistent* with the governing partial differential equation.

[10 marks]

3. For the following finite difference schemes:

(a) Draw the stencil associated with the scheme.

(b) Trace the dependence of u_m^3 on the previous time levels.

(c) State the dependence of u_m^3 on the initial time ($n = 0$).

(i) FTFS:

$$u_m^{n+1} = (1 + a\lambda)u_m^n - a\lambda u_{m+1}^n, \quad a, \lambda \in \mathbb{R}.$$

[2,5,3 marks]

(ii) Lax-Friedrichs:

$$u_m^{n+1} = \frac{1}{2}(1 - a\lambda)\, u_{m+1}^n + \frac{1}{2}(1 + a\lambda)\, u_{m-1}^n, \quad a, \lambda \in \mathbb{R}.$$

[2,5,3 marks]

4. A boundary value problem is given by

$$u''(x) + 2u'(x) = f(x), \quad u(4) = 1, \quad u'(0) = 2, \quad 0 \le x \le 4.$$

By introducing an arbitrary weight function $w(x) \in \mathcal{H}^1$ such that $w(4) = 0$, determine and state the weak (variational) form of this boundary value problem. [10 marks]

5. The equation

$$au_t + u_x = u, \quad a \in \mathbb{R},$$

is approximated by the following finite difference scheme

$$\frac{a}{k}\left[u_m^{n+1} - \frac{1}{2}\left(u_{m+1}^n + u_{m-1}^n\right)\right] + \frac{1}{2h}\left[u_{m+1}^n - u_{m-1}^n\right] = u_m^n.$$

Show that, provided the ratio $\mu = h/k$ is kept constant as $h, k \to 0$, then the scheme has accuracy of order 1 for both space and time. [15 marks]

Section B

6. Investigate the stability of the following finite difference scheme

$$u_m^{n+1} = \alpha u_m^n + (1-\alpha) u_{m-1}^n, \quad \alpha \in \mathbb{R}.$$

Using the discrete energy norm $||.||_h^2$ and inequalities given in the supplementary material at the end of the paper:

(a) State any conditions on the constant $\alpha \in \mathbb{R}$ necessary to ensure the system is stable for finite n. Justify your answer. [8 marks]

(b) Determine whether there are any values of α for which the system is stable in the limit $n \to \infty$. Justify your answer. [4 marks]

7. Consider the following boundary value problem

$$u'' + 2x^2 = 0, \quad u(1) = 1, \quad u'(0) = 0, \quad 0 \leq x \leq 1.$$

The weak form of this boundary value problem is given by

$$\int_0^1 u'(x)\, w'(x)\, dx = 2\int_0^1 x^2 w(x)\, dx,$$

where $w(x) \in \mathcal{H}^1$ is an arbitrary weight function such that $w(1) = 0$.

Assume that the solution $u(x)$ and arbitrary weight function $w(x)$ are approximated by

$$u^h(x) = u_0\phi_0(x) + u_1\phi_1(x) + \phi_2(x), \quad w^h(x) = w_0\phi_0(x) + w_1\phi_1(x),$$

where the shape functions ϕ_i take the form

$$\phi_0(x) = \begin{cases} 1-2x, & 0 \leq x \leq \frac{1}{2}, \\ 0, & \frac{1}{2} \leq x \leq 1, \end{cases}, \quad \phi_1(x) = \begin{cases} 2x, & 0 \leq x \leq \frac{1}{2}, \\ 2(1-x), & \frac{1}{2} \leq x \leq 1, \end{cases},$$

$$\phi_2(x) = \begin{cases} 0, & 0 \leq x \leq \frac{1}{2}, \\ 2x-1, & \frac{1}{2} \leq x \leq 1. \end{cases}$$

(a) Determine the Galerkin equation associated with this boundary value problem, in the form

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}.$$

[18 marks]

(b) Solve the Galerkin equation to obtain the constants u_1 and u_2. State the final expression for $u^h(x)$. [8 marks]

8. The first order partial derivatives of a function $u(x,t)$ can be approximated by the finite differences

$$u_x = \frac{1}{h}\Big(u(x+h,t) - u(x,t)\Big) = \frac{1}{h}\left(u^n_{m+1} - u^n_m\right),$$

$$u_t = \frac{1}{2k}\Big(u(x,t+k) - u(x,t-k)\Big) = \frac{1}{2k}\left(u^{n+1}_m - u^{n-1}_m\right).$$

Using the schemes given above, or otherwise, derive a finite difference scheme capable of modeling the third order partial differential equation

$$u_{xxx} - a u_{tt} = 0, \quad a \in \mathbb{R}.$$

[12 marks]

Supplementary material

For an infinitely differentiable function $\phi(x,t)$, the Taylor expansion in x about a fixed point (x,t) is given by

$$\phi(x+h,t) = \phi(x,t) + h\phi_x(x,t) + \frac{h^2}{2!}\phi_{xx}(x,t) + \frac{h^3}{3!}\phi_{xxx}(x,t) + \frac{h^4}{4!}\phi_{xxxx}(x,t) + O(h^5),$$

where $|h| \ll 1$. Similarly, the Taylor expansion for t about a fixed point (x,t) takes the form

$$\phi(x,t+k) = \phi(x,t) + k\phi_t(x,t) + \frac{k^2}{2!}\phi_{tt}(x,t) + \frac{k^3}{3!}\phi_{ttt}(x,t) + \frac{k^4}{4!}\phi_{tttt}(x,t) + O(k^5),$$

where $|k| \ll 1$ is a small constant.

The discrete energy norm $\|.\|^2_h$ of a function $v_m(x)$ is given by

$$\|v_m(x)\|^2_h = h \sum_{m=-\infty}^{\infty} |v_m(x)|^2 .$$

The following inequalities may be assumed without proof

$$|x+y| \le |x|^2 + 2|x||y| + |y|^2, \quad 2|x||y| \le |x|^2 + |y|^2.$$