PRIFYSGOL ABERYSTWYTH UNIVERSITY

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 2 / SEMESTER 2 EXAMINATIONS

MAI / MAY 2020

MA34810 - Mathematical Models of Biological Systems

The questions on this paper are written in English.

If you have questions about the paper during the exam, contact the module co-ordinator, Simon Cox, on sxc@aber.ac.uk.

You should write out solutions to the paper and upload them to blackboard as a single PDF file.

Amser a ganiateir - 3 awr
Maen rhaid cyflwyno eich atebion erbyn 12:30yb (amser y DU).

- Gellir rhoi cynnig ar bob cwestiwn.
- Rhoddir mwy o ystyriaeth i berfformiad yn rhan B wrth bennu marc dosbarth cyntaf.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

Time allowed - 3 hours
Submission must be completed by 12:30pm (UK time).

- All questions may be attempted.
- Performance in section B will be given greater consideration in assigning a first class mark.
- Students may submit answers to this paper in either Welsh or English.

Section A

1. A model for the growth of a population is

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \rho N\left(1 - N\right)$$

where $\rho > 0$.

(a) Explain what the parameter ρ might represent in a biological system. [1 mark]

(b) Find all equilibria and examine their stability. [6 marks]

2. Construct a cobweb diagram for each of the following maps, and hence determine the qualitative global behaviour of the solutions for initial populations N_0 between 0 and 1.

(a)

$$N_{t+1} = N_t^2.$$

[5 marks]

(b)

$$N_{t+1} = rN_t(1 - N_t^2),$$

for $r > 1$. [5 marks]

3. Consider the discrete map

$$N_{t+1} = g(N_t)$$

for some function g and a population N_t.

(a) Give the definition of an $m-$periodic steady state N^* of g. [2 marks]

(b) Find both steady states of this map if g is the function

$$g(N) = rN(2 - N),$$

where $r > 0$. [2 marks]

(c) Find the value of r for which there is a bifurcation to period 2 solutions. [3 marks]

4. For the following system of differential equations

$$\dot{x} = 2x + 5y,$$
$$\dot{y} = -4x - 7y.$$

(a) Find the steady state and determine its stability. [4 marks]

(b) Find the general solution of this system. [6 marks]

(c) Sketch the phase plane diagram. [4 marks]

5. For the following system of differential equations

$$\dot{x} = x - 5y,$$
$$\dot{y} = x - 3y.$$

(a) Find the steady state and determine its stability. [4 marks]

(b) Find the general solution of this system. [6 marks]

(c) Sketch the phase plane diagram. [4 marks]

6. (a) Describe briefly a biological system that this reaction-diffusion scheme might be modelling if $r > 0, c_1 < -1, -1 < c_2 < 0$: [2 marks]

$$\frac{\partial u}{\partial t} = u(1 - u + c_1 v) + D_1 \frac{\partial^2 u}{\partial x^2},$$
$$\frac{\partial v}{\partial t} = rv(1 + c_2 u - v) + D_2 \frac{\partial^2 v}{\partial x^2}.$$

(b) In the spatially homogeneous case ($D_1 = D_2 = 0$) the equilibrium points of the scheme are

$$(u^*, v^*) = (0,0), (1,0), (0,1), \left(\frac{1 + c_1}{1 - c_1 c_2}, \frac{1 + c_2}{1 - c_1 c_2} \right).$$

By examining the existence and stability of the equilibria at which u and v are not both zero, determine the behaviour of the system for initial conditions $u > 0, v > 0$. [10 marks]

7. Consider the following chemical reaction scheme:

$$A \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} X, \qquad B \xrightarrow{k_2} Y, \qquad 2X + Y \xrightarrow{k_3} 3X.$$

The concentrations a and b, of A and B respectively, are kept constant. State the Law of Mass Action and use it to derive a pair of differential equations for the concentrations of X and Y. [6 marks]

Section B

8. Consider the following model for the harvesting of a population $N(t)$ with constant effort E:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = aN\ln\left(\frac{K}{N}\right) - EN,$$

for positive constants a and K.

(a) Describe the biological meaning of the parameter K. [2 marks]

(b) In the absence of harvesting ($E = 0$), find the possible equilibrium populations for N and determine their stability. [5 marks]

(c) Find the equilibrium solutions of this system for $E > 0$. [4 marks]

(d) Determine the stability of any positive equilibria. [4 marks]

9. Consider the following model for the dynamics of a directly transmitted parasite, in which S is the number of susceptibles, I the number of infectives and M the number of immunes:

$$\begin{aligned}\dot{S} &= bN - \beta SI - bS\\ \dot{I} &= \beta SI - (b+r)I\\ \dot{M} &= rI - bM.\end{aligned}$$

The constants b, r and β are all positive.

(a) Show that the total population $S + I + M = N$ is constant. [4 marks]

(b) Find the equilibria and examine their stability. [15 marks]

(c) Show that there is a threshold population size N_c such that if $N < N_c$ the parasite cannot maintain itself and both the infectives and the immunes eventually die out. [2 marks]

10. Consider the following chemical reaction scheme:

$$A + Y \xrightarrow{k_1} X + P, \qquad X + Y \xrightarrow{k_2} 2P, \qquad A + X \xrightarrow{k_3} 2X + Y, \qquad 2X \xrightarrow{k_4} A + P,$$

in which the concentration α of A is kept constant.

(a) What is meant by an *autocatalytic* reaction? State which of these reactions are autocatalytic for X and/or Y. [2 marks]

(b) Use the Law of Mass Action to derive a pair of differential equations for the concentrations of X and Y. [4 marks]

(c) Give an appropriate non-dimensionalization, and hence show that these equations can be written in the form

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \beta_1 v - uv + u - \beta_2 u^2, \quad \frac{\mathrm{d}v}{\mathrm{d}t} = -\beta_1 v - uv + u.$$

You should express β_1 and β_2 in terms of the parameters k_1, k_2, k_3, k_4 and α. [8 marks]