ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 2 / SEMESTER 2 EXAMINATIONS

MAI / MAY 2020

MA37410 - Probability and Stochastic Processes

The questions on this paper are written in English.

If you have questions about the paper during the exam, contact the module co-ordinator, Dr Rolf Gohm, on rog@aber.ac.uk.

You should write out solutions to the paper and upload them to blackboard as a single PDF file.

Amser a ganiateir - 3 awr
Mae'n rhaid cyflwyno eich atebion erbyn 12:30 (amser y DU).

- Gellir rhoi cynnig ar bob cwestiwn.
- Rhoddir mwy o ystyriaeth i berfformiad yn rhan B wrth bennu marc dosbarth cyntaf.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

Time allowed - 3 hours
Submission must be completed by 12:30 (UK time).

- All questions may be attempted.
- Performance in section B will be given greater consideration in assigning a first class mark.
- Students may submit answers to this paper in either Welsh or English.

Section A

1. Suppose that X and Y have joint probability density function (pdf)

$$f(x,y) = A\,x^2,\ 0 < y < x < 2$$

(and 0 elsewhere). A is a positive constant.

(a) Sketch the region where $f(x,y)$ is positive and find the marginal densities of X and Y. [6 marks]

(b) Determine the value of the constant A so that $f(x,y)$ is indeed a pdf. [2 marks]

(c) Calculate the expectations $\mathbb{E}X$ and $\mathbb{E}Y$. [5 marks]

(d) Find the conditional density function $f_{Y|X=x}(y)$ and the conditional expectation $\mathbb{E}(Y|X=x)$. [4 marks]

(e) Verify that $\mathbb{E}\mathbb{E}[Y|X] = \mathbb{E}Y$ [3 marks]

2. Let X be a random variable with a normal distribution $N(-7, 22)$.

(a) Find the distribution of $Y = 2X + 3$. [4 marks]

(b) Find the distribution of $Z = 5(X_1 + X_2)$, where X_1, X_2 are independent and both have the same distribution as X. [6 marks]

3. (a) State the Central Limit Theorem for an i.i.d.-sequence $X_1, X_2, \ldots$ of random variables. What does the abbreviation i.i.d. mean? [4 marks]

(b) Let $X_1, X_2, \ldots, X_{100}$ be independent random variables, each with a uniform distribution in the interval [-1,1]. Make use of the Central Limit Theorem to find an approximation of the probability that $\sum_{i=1}^{100} X_i$ is bigger than 10. [6 marks]

4. The number of customers is a random variable N with mean 20 and variance 100. The i-th customer spends an amount A_i, which has mean 15 and variance 25. The A_i are independent of each other and of N. If the total amount spent is $T = A_1 + A_2 + \cdots + A_N$, find

(a) $\mathbb{E}[T|N]$, [2 marks]

(b) Var$[T|N]$ [3 marks]

(c) and hence Var$[T]$. [4 marks]

For full marks include short justifications of your reasoning.

5. Consider a branching process with a single ancestor in generation zero and with a family size distribution p_k (for k the number of offspring of an individual) given by $p_0 = 0.2,\ p_1 = 0.4,\ p_2 = 0.3,\ p_3 = 0.1$.

(a) What is the pgf of this distribution? What is the expected family size? [3 marks]

(b) Calculate the probability that the process eventually dies out. State the result about branching processes which you use for your calculation. [6 marks]

6. (a) State the Markov property (of a Markov chain). [3 marks]

(b) Consider the transition matrix

$$P = \begin{pmatrix} 1/3 & 0 & 2/3 \\ 1/4 & 1/2 & 1/4 \\ 1/8 & 3/4 & 1/8 \end{pmatrix}.$$

Labeling the states a, b and c (top to bottom, left to right), what is the probability that

(i) if $X_0 = c$ then $X_1 = a$?

(ii) if $X_0 = c$ then $X_1 = a$ and then $X_2 = a$?

(iii) if $X_0 = c$ then $X_2 = a$? ? [4 marks]

(c) Find a stationary distribution for the transition matrix in (b). [5 marks]

Section B

7. (a) For a sequence of random variables state the definitions of convergence in probability and convergence in distribution. [4 marks]

(b) Give an example of a sequence which is convergent in distribution but not convergent in probability. Include a short argument why this is the case. [6 marks]

8. (a) Let $T = \sum_{i=1}^{R} Z_i$, where R and the Z_i are independent random variables taking values in the non-negative integers (and $R \geq 1$). . Say R has pgf G_R and the Z_i are identically distributed, each with pgf G. Show that for the pgf G_T of T we have

$$G_T(s) = G_R(G(s)).$$

[5 marks]

(b) State the definition of a branching process $(X_n)_{n\geq 0}$. [3 marks]

(c) Assuming that $X_0 = 1$ show with (a) and (b) that for the pgf G_n of X_n we have

$$G_n(s) = G \circ G \circ \ldots \circ G \qquad (n \text{ factors}),$$

where $\circ$ denotes composition and G is the pgf of the family size distribution. [4 marks]

(d) Prove from (c) that the probability π of ultimate extinction for a branching process is a fixed point of G. [5 marks]

9. A Markov chain has the transition matrix $P = (p_{ij})_{i,j=1,\ldots,6}$

$$\begin{bmatrix} 0.4 & 0.1 & 0 & 0.5 & 0 & 0 \\ 0 & 0.9 & 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0.8 & 0 & 0 & 0.2 \\ 0 & 0 & 0.4 & 0.3 & 0 & 0.3 \\ 0 & 0.5 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0.6 & 0.4 & 0 & 0 \end{bmatrix}.$$

(a) Find the communicating classes of states and classify them as recurrent or transient. [4 marks]

(b) Compute $\lim_{n\to\infty} p_{12}^{(n)}$
(the long-term transition probability from state 1 to state 2). [7 marks]

10. Initially box A contains 1 red ball and 2 blue balls and box B contains 3 red balls. Now in each step we choose randomly one of the balls in each box (with equal probability and independently from each other) and interchange them. Note that by performing these steps the number of red balls in box A may change but there are always three balls in box A.

(a) Model the situation above by a Markov chain and determine the transition matrix. [6 marks]

(b) What is the expected number of steps until we have 3 red balls in box A for the first time? [6 marks]