PRIFYSGOL ABERYSTWYTH UNIVERSITY

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 2 / SEMESTER 2 EXAMINATIONS

MAI / MAY 2020

MA38010 - Spectral Theory

The questions on this paper are written in English.

If you have questions about the paper during the exam, contact the module co-ordinator, Dr Rob Douglas, on rsd@aber.ac.uk.

You should write out solutions to the paper and upload them to Blackboard as a single PDF file.

Amser a ganiateir - 3 awr
Mae'n rhaid cyflwyno eich atebion erbyn 12:30 (amser y DU).

- Gellir rhoi cynnig ar bob cwestiwn.
- Rhoddir mwy o ystyriaeth i berfformiad yn rhan B wrth bennu marc dosbarth cyntaf.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

Time allowed - 3 hours
Submission must be completed by 12:30 (UK time).

- All questions may be attempted.
- Performance in section B will be given greater consideration in assigning a first class mark.
- Students may submit answers to this paper in either Welsh or English.

Section A

1. (a) Let $(X, \langle ., . \rangle)$ be an inner product space.

(i) Explain what is meant by saying that x and y are *orthogonal*, where $x, y \in X$. [2 marks]

(ii) Let $\|.\|$ denote the induced norm on X. Suppose that $u, v \in X$ are such that $\|u + 2v\| = 5$, $\|u - 2v\| = 9$, and $\|u\| = 7$. Determine $\|v\|$. [4 marks]

(b) Let $\omega = e^{2\pi i/3}$.

(i) Demonstrate that $(\omega^2, 1, \omega)$ and $(\omega, 1, \omega^2)$ are orthogonal vectors in $\mathbb{C}^3$, with respect to the standard inner product on $\mathbb{C}^3$. [3 marks]

(ii) Find the vector in sp$((\omega^2, 1, \omega), (\omega, 1, \omega^2))$ that is nearest to $(3, 3, -3)$. [6 marks]

2. Let W be the subspace of $\mathbb{R}^4$ defined by

$$W = \text{sp}((1, 2, 2, 3), (1, -2, 0, 1), (1, 0, 1, 0)).$$

Use Gram-Schmidt orthogonalisation to find an orthogonal basis $\{u_1, u_2, u_3\}$ for W, with respect to the standard inner product on $\mathbb{R}^4$. [6 marks]

3. Suppose that $M = \text{sp}((2, 0, 2, 0), (0, 3, 0, 0)) \subset \mathbb{R}^4$.

(a) Find $M^\perp$, the orthogonal complement of M (with respect to the standard inner product on $\mathbb{R}^4$). [4 marks]

(b) Show that $M \oplus M^\perp = \mathbb{R}^4$, where $\oplus$ denotes direct sum. [4 marks]

4. Let $l^2_{\mathbb{C}}$ denote the vector space of square summable complex sequences. Define $B : l^2_{\mathbb{C}} \to l^2_{\mathbb{C}}$ by

$$B(x) = (x(3), x(4), x(5), ...),$$

where $x = (x(1), x(2), x(3), ...)$.

(a) Demonstrate that B is linear and bounded; calculate $\|B\|$, the operator norm of B. [2,3,2 marks]

(b) Show that B is surjective, but not injective. [2,2 marks]

(c) Calculate B^*, the adjoint of B. [2 marks]

5. Let $l^2_{\mathbb{R}}$ denote the vector space of square summable real sequences. Define $T : l^2_{\mathbb{R}} \to l^2_{\mathbb{R}}$ by

$$T(x) = (x(1)/8, x(2)/11, ..., x(n)/(n^2 + 7), ...).$$

(a) Show that T is linear and bounded; calculate $\|T\|$, the operator norm of T. [2,3,2 marks]

(b) Demonstrate that T is self-adjoint. [2 marks]

(c) Prove that T is compact. [7 marks]

(You may state any results you require about compact operators without proof.)

(d) Calculate the eigenvalues of T. [4 marks]

6. (a) Let H be a Hilbert space, and suppose that $A \in \mathcal{B}(H)$, the vector space of bounded linear mappings from H to itself. What is meant by the *spectrum* of A, denoted $\sigma(A)$, and the *point spectrum* of A, denoted $\sigma_p(A)$? Demonstrate that $\sigma_p(A) \subset \sigma(A)$. [2,2 marks]

(b) Let $A \in \mathcal{B}(l^2_{\mathbb{C}})$ be defined by

$$A(x) = (x(1)/3, x(2)/6, ..., x(n)/(n^2 + 2), ...).$$

Demonstrate that $0 \in \sigma(A) \backslash \sigma_p(A)$. [4 marks]

Section B

7. Let $\varphi_1, \varphi_2, \varphi_3$ be non-zero orthogonal vectors in an inner product space $(X, \langle ., . \rangle)$ of dimension 3.

(a) Show that the set $\{\varphi_1, \varphi_2, \varphi_3\}$ is linearly independent. [3 marks]

(b) Prove that for $x \in X$,

$$x = \sum_{i=1}^{3} \langle x, \varphi_i \rangle \varphi_i / \|\varphi_i\|^2,$$

where $\|.\|$ denotes the norm induced by the inner product. [3 marks]

8. Let $(X, \langle ., . \rangle)$ be a real inner product space. Let $(x_n)_{n=1}^{\infty} \subset X$, $x \in X$, and suppose that $\langle x_n, y \rangle \to \langle x, y \rangle$ as $n \to \infty$ for every $y \in X$, and that $\|x_n\| \to \|x\|$ as $n \to \infty$ where $\|.\|$ denotes the norm induced by the inner product. Prove that $x_n \to x$ as $n \to \infty$. [5 marks]

9. Let H be a Hilbert space, and suppose $(u_n)_{n=1}^{\infty}$ is a complete orthonormal sequence in H. For $x \in H$, prove that $x - \sum_{k=1}^{n} a_k u_k$ is orthogonal to all linear combinations of the form $\sum_{k=1}^{n} b_k u_k$, where $b_1, ..., b_k$ are scalars, if and only if $a_k = \langle x, u_k \rangle$ for $k = 1, ..., n$. [8 marks]

10. (a) Let X be a Banach space, and suppose $T \in \mathcal{B}(X)$ is such that $\|T\| = 1/2$, where $\mathcal{B}(X)$ denotes the vector space of bounded linear mappings from X to itself. Write $S = \sum_{n=0}^{\infty} T^n$, where T^0 is the identity mapping I. Prove that $(I - T) \circ S = I$, where $\circ$ denotes composition of functions. [8 marks]

(You may use the result that an absolutely convergent series in a Banach space is convergent without proof.)

(b) For $A, B \in \mathcal{B}(l^2_{\mathbb{R}})$, where $l^2_{\mathbb{R}}$ denotes the vector space of square summable real sequences, does $A \circ B = I$ imply A and B are invertible? Give a proof or counterexample as appropriate. [3 marks]

11. Let H be a Hilbert space, and let $T \in \mathcal{B}(H)$. Suppose T is a rank two operator, that is its image space is 2-dimensional. Let Ψ_1, Ψ_2 be non-zero orthogonal vectors in the image space of T. Show that there exist $\varphi_1, \varphi_2 \in H$ such that

$$T(x) = \langle x, \varphi_1 \rangle \Psi_1 + \langle x, \varphi_2 \rangle \Psi_2$$

for every $x \in H$. [5 marks]

12. Let H be a complex Hilbert space.

(a) Let $A_1, A_2 \in \mathcal{B}(H)$, $\alpha, \beta \in \mathbb{C}$. Demonstrate that that $(\alpha A_1 + \beta A_2)^* = \overline{\alpha} A_1^* + \overline{\beta} A_2^*$, where A^* denotes the adjoint of $A \in \mathcal{B}(H)$. [5 marks]

(b) Let $T \in \mathcal{B}(H)$, $S_1, S_2 \subset H$. Suppose $T(S_1) \subset S_2$. Prove that $T^*(S_2^{\perp}) \subset S_1^{\perp}$, where $S^{\perp}$ denotes the orthogonal complement of $S \subset H$. [4 marks]

13. Let H be an infinite dimensional Hilbert space, and suppose $A \in \mathcal{B}(H)$ satisfies $A^5 = I$, where I denotes the identity mapping. Demonstrate that A^2 is not compact. [6 marks]