PRIFYSGOL ABERYSTWYTH UNIVERSITY

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 2 / SEMESTER 2 EXAMINATIONS

MAI / MAY 2020

MA38310 – Operator Algebra

The questions on this paper are written in English.

If you have questions about the paper during the exam, contact the module co-ordinator, Dr Gwion Evans, on dfe@aber.ac.uk.

You should write out solutions to the paper and upload them to blackboard as a single PDF file.

Amser a ganiateir - 3 awr
Mae'n rhaid cyflwyno eich atebion erbyn 12:30 (amser y DU).

- Gellir rhoi cynnig ar bob cwestiwn.
- Rhoddir mwy o ystyriaeth i berfformiad yn rhan B wrth bennu marc dosbarth cyntaf.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

Time allowed - 3 hours
Submission must be completed by 12:30 (UK time).

- All questions may be attempted.
- Performance in section B will be given greater consideration in assigning a first class mark.
- Students may submit answers to this paper in either Welsh or English.

Section A

1. Consider the quaternions $q_1 = 8 - 2i + 4j - 6k$ and $q_2 = -1 - 3i + 3j + 3k$. Write the following in the form $w + xi + yj + zk$ for some $w, x, y, z \in \mathbb{R}$:

(a) $3q_1 + 2q_2$; (b) $q_1 q_2$; (c) $q_1 q_2 - q_2 q_1$; (d) q_1^{-1}.

[11 marks]

2. For each of the following equations, find its solution set in $\mathbb{H}$ and, if the set is infinite, state 7 explicit solutions in the form $w + xi + yj + zk$ for some $w, x, y, z \in \mathbb{R}$.

(a) $6x = x^2 + 25$; (b) $25x^2 + 30x + 8 = 0$; (c) $2x^2 = 4x - 5$.

[10 marks]

3. Consider $M_3(\mathbb{Q})$ as an algebra over $\mathbb{Q}$ in the usual way.

Determine which of the following subsets of $M_3(\mathbb{Q})$ are subalgebras and which are unital subalgebras. Justify your answers clearly.

(a) $S_1 = \left\{ \begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{pmatrix} : a, b, c, d, e \in \mathbb{Q} \right\}$; (c) $S_3 = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in \mathbb{Q} \right\}$.

(b) $S_2 = \left\{ \begin{pmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & 0 & 0 \end{pmatrix} : a, b, c \in \mathbb{Q} \right\}$;

[12 marks]

4. Consider $M_2(\mathbb{C})$ and $\mathbb{C}$ as algebras over $\mathbb{C}$ in the usual way.

Determine which of the following mappings are homomorphisms. For those that are, determine whether or not they are isomorphisms. Justify your answers clearly.

(a) $\pi_1 : M_2(\mathbb{C}) \to \mathbb{C}, \qquad \pi_1\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = b.$

(b) $\pi_2 : M_2(\mathbb{C}) \to M_2(\mathbb{C}), \qquad \pi_2\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right) = \begin{pmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{pmatrix}.$

[10 marks]

5. Recall that the left regular representation of an algebra A is the homomorphism $\rho : A \to \mathcal{L}(A)$, from A into the algebra of linear transformations on A, given by $\rho(a)(b) = ab$ for all $a, b \in A$. Denote by $\mu_{\mathcal{B}}$ the matrix representation homomorphism of $\mathcal{L}(A)$ with respect to a basis $\mathcal{B}$.

You may assume, without proof, that

$$A := \left\{ \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix} : a_{11}, a_{21}, a_{22} \in \mathbb{C} \right\}$$

is an algebra over $\mathbb{C}$ (with operations inherited from $M_2(\mathbb{C})$). Let

$$f_1 := \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad f_2 := \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \quad f_3 : \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and let $\mathcal{B} := \{f_1, f_2, f_3\}$. You may assume without proof that:

$$\begin{aligned} f_1 f_1 &= f_1 & f_1 f_2 &= f_1 & f_1 f_3 &= \mathbf{0} \\ f_2 f_1 &= f_2 & f_2 f_2 &= f_2 & f_2 f_3 &= \mathbf{0} \\ f_3 f_1 &= \frac{1}{2} f_1 - \frac{1}{2} f_2 & f_3 f_2 &= -\frac{1}{2} f_1 + \frac{1}{2} f_2 & f_3 f_3 &= f_3 \end{aligned} \qquad (\star)$$

(a) Prove that $\mathcal{B}$ is a basis for A. [6 marks]

(b) Let $a \in A$. By writing $a = \alpha f_1 + \beta f_2 + \gamma f_3$ $(\alpha, \beta, \gamma \in \mathbb{C})$, and using the multiplicative relations $(\star)$, find $\mu_{\mathcal{B}} \circ \rho(a)$, i.e., the matrix representation of $\rho(a)$ with respect to the basis $\mathcal{B}$. [6 marks]

6. Let $v_1 = (1, 0, i)$, $v_2 = (0, 2i, -i)$ and let $M = \operatorname{span}\{v_1, v_2\}$ in the Hilbert space $\mathbb{C}^3$.

(a) Use the Gram-Schmidt process to find an orthonormal basis for M. [5 marks]

(b) Find the matrix representation of the orthogonal projection of $\mathbb{C}^3$ onto M with respect to the standard basis of $\mathbb{C}^3$. [5 marks]

7. Consider the self-adjoint element $a = \begin{pmatrix} -2 & 2i \\ -2i & 1 \end{pmatrix}$ in the C^*-algebra $M_2(\mathbb{C})$.

(a) Find the spectrum of a, denoted $\sigma(a)$. [3 marks]

(b) Find the norm of a, denoted $\|a\|$. [2 marks]

Section B

8. (a) Recall the unitization of an algebra C over $\mathbb{C}$, denoted by $\widetilde{C}$ and given, as a set, by

$$\widetilde{C} := \{(x, \lambda) : x \in C,\ \lambda \in \mathbb{C}\}.$$

Show that the map $\phi : \widetilde{C} \to \mathbb{C}$ given by $\phi((x, \lambda)) = \lambda$ is a homomorphism.

Hint: be careful when recalling the algebra operations on $\widetilde{C}$, especially multiplication. [4 marks]

(b) Given that

$$A := \left\{ \begin{pmatrix} \lambda & a_{12} & a_{13} \\ 0 & \lambda & a_{23} \\ 0 & 0 & \lambda \end{pmatrix} : \lambda, a_{ij} \in \mathbb{C} \text{ for all } i, j \right\}$$

is an algebra (with operations inherited from $M_3(\mathbb{C})$), show that

$$B := \left\{ \begin{pmatrix} 0 & b_{12} & b_{13} \\ 0 & 0 & b_{23} \\ 0 & 0 & 0 \end{pmatrix} : b_{ij} \in \mathbb{C} \text{ for all } i, j \right\}$$

is an ideal of A. [4 marks]

(c) Prove that A is isomorphic to the unitization of B, by defining an appropriate map $\psi : \widetilde{B} \to A$ and showing it to be an isomorphism.

Hint: note that $a \in A$ can be written as $b + \lambda \mathbb{1}$ for some matrix b and some scalar λ, where $\mathbb{1}$ is the identity matrix. [4 marks]

(d) Find a surjective homomorphism $\pi : A \to \mathbb{C}$ and apply the First Isomorphism Theorem to prove that the quotient algebra A/B is isomorphic to $\mathbb{C}$. [6 marks]

9. Suppose that $A = M_2(\mathbb{C})$, the C^*-algebra of 2×2 matrices with complex entries. Let

$$a = \begin{pmatrix} 8 & -2i \\ 2i & 5 \end{pmatrix}.$$

(a) Verify that $\sigma(a) \subseteq [0, \infty)$ by calculating $\sigma(a)$. [3 marks]

(b) Determine the spectral projections of a. [4 marks]

(c) For each $\lambda \in \sigma(a)$, let $\chi_\lambda : \sigma(a) \to \mathbb{C}$ be the characteristic function of λ, i.e.,

$$\chi_\lambda(t) = \begin{cases} 1 & \text{if } t = \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Let $f : \sigma(a) \to \mathbb{C}$ be the square root function, i.e., $f(t) = \sqrt{t}$.
Write f as a linear combination of $\chi_\lambda,\ \lambda \in \sigma(a)$. [2 marks]

(d) Hence, find a square root of a, i.e., a matrix $c \in M_2(\mathbb{C})$ such that $a = c^2$.[4 marks]

10. (a) Let $a = (a_{ij})_{i,j=1}^n$ be a diagonal matrix in the $*$-algebra $M_n(\mathbb{C})$. Prove that

$$\{a\}' = \{(x_{ij})_{i,j=1}^n \in M_n(\mathbb{C}) : (a_{ii} - a_{jj})x_{ij} = 0 \text{ for all } i, j\}.$$

[4 marks]

(b) Find the commutants of the following subsets in the given algebras:

(i) $\left\{ \begin{pmatrix} i & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \subseteq M_3(\mathbb{C})$, (ii) $\left\{ \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \right\} \subseteq M_4(\mathbb{C})$.

[5 marks]

(c) Let S, T be non-empty subsets of an algebra A. Prove that

(i) $S \subseteq S''$,

(ii) if $S \subseteq T$ then $T' \subseteq S'$.

[4 marks]

(d) Let S be a non-empty self-adjoint subset of $B(H)$, i.e., the $*$-algebra of bounded linear operators on a Hilbert space H. Prove that S' is a von Neumann algebra. [3 marks]

(e) Which, if any, of the commutants in part (b) are von Neumann algebras? Justify your answer briefly. [3 marks]