PRIFYSGOL ABERYSTWYTH UNIVERSITY

**ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS**

**ARHOLIADAU SEMESTER 1 / SEMESTER 1 EXAMINATIONS**

**IONAWR / JANUARY 2026**

# MA10310 – Probability

The questions on this paper are written in English.

**Amser a ganiateir - 2 awr**

- Rhoddir marciau llawn am atebion cyflawn i bob cwestiwn yn Rhan A ac i dri cwestiwn yn Rhan B.
- Yn Rhan B, rhoddir credyd am y tri ateb gorau.
- Cyfrifianellau Casio FX-83 neu FX-85 YN UNIG a ganiateir.
- Darperir tablau ystadegol.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

**Time allowed - 2 hours**

- Full marks will be given for complete answers to all questions in Section A and to three questions in Section B.
- In Section B, credit will be given for the best three answers.
- Casio FX-83 or FX-85 calculators ONLY may be used.
- Statistical tables will be provided.
- Students may submit answers to this paper in either Welsh or English.

© 2026 Aberystwyth University

## Section A

**1**. You are giving an after dinner speech and you ask a friend for some jokes. They give you a list of jokes, of which 5 are funny, 3 are not funny and 2 are just inappropriate. Say you choose 3 jokes at random from the list, find the probability that:

(a) all of them are funny;

(b) you make at least one inappropriate joke. [5 marks]

**2**. State the (Kolmogorov) Axioms of Probability. Prove **directly** from these axioms, using event laws as required, that for some generic events $A, B$,

$$\mathbb{P}\Big(A^{\complement} \cap B\Big) = \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

[5 marks]

**3**. You will finish eating your lunch at your favourite picnic bench any time between 12:30 and 1:40 PM (with all times equally likely between). The most aggressive seagull on campus will start patrolling around your bench any time between 1:00 and 1:27 (again, all times equally likely between). The seagull is certain to make a dive and attempt to snatch your food if it arrives while you are eating. Assuming that the end time of your lunch and the arrival time of the seagull are independent, find the probability that you eat your lunch without being challenged for your food by the seagull. [5 marks]

**4**. You are taking bets on a horse race for your friends. You offer odds of:

| **Horse:** | Fat Chance | Lucky Inmate | Monty Haul | Die Loaded |
|---|---|---|---|---|
| **Odds:** | 14/1 | 7/2 | 3/2 on | 8/1 |

[Odds are against, unless specified as "on", when they are in favour.]

(a) Show that you are offering *fair* odds.

(b) Assuming that the odds reflect the probability of winning, find the probability that the favourite wins. [5 marks]

**5**. A standard $n$-sided die has faces labelled $1, 2, \ldots, n$, with each outcome equally likely. Consider a game where you win £$3r$ if an $n$-sided die lands on side $r$. If the *expected* winnings are £19.5, then find $n$.
[ HINT: $\sum_{r=1}^{n} r = n(n+1)/2$. ] [5 marks]

**6**. The probability that a true prince will arrive and wake Sleeping Beauty in any given year is $1/70$. Given that she has slept for 50 years already, find the probability that she will be awakened within the next 50 years. [5 marks]

**7**. The duration, $X$ minutes, of a song by the Piano Man has probability density function

$$f(x) = \begin{cases} 0, & x < 3; \\ \frac{1}{2}\mathrm{e}^{(3-x)/2}, & x \geq 3. \end{cases}$$

Find

(a) the cumulative distribution function of $X$. [3 marks]

(b) Assuming he starts a song at 9:02 PM, find the probability that it finishes after 9:06 PM. [2 marks]

**8**. Let $X$ have cumulative distribution function

$$F(x) = \begin{cases} 0, & x < 0; \\ 3x^2 - 2x^3, & 0 \leq x \leq 1; \\ 1, & x > 1. \end{cases}$$

(a) Find the probability density function of $X$;

(b) and deduce the expected value of $X$. [5 marks]

## Section B

**9**. (a) You are searching for your lost cat. She is in one of the attic, the bedroom, the kitchen or the garden. You assume that she is equally likely to be in the bedroom or kitchen, twice as likely to be in the bedroom than the attic, and there is a 50% chance she is in the garden.

Your mother, who adores the cat and left strict instructions for you to keep an eye on her, will soon return. If the cat is in the garden then there is only a 5% chance you will find her in time, whereas for the attic, bedroom and kitchen the chances are 20%, 70% and 60% respectively.

(i) Find the probabilities of her being in each of the locations. [2 marks]

(ii) Draw a suitable diagram to represent the probabilities of finding the cat. [4 marks]

Find the probability that:

(iii) you find the cat in the kitchen; [1 mark]

(iv) you don't find the cat in time; [2 marks]

(v) the cat was in the bedroom given that you find her. [2 marks]

(b) You hear a meow and you are certain it is from within the house. Assuming the relative probabilities of the remaining possible rooms stay the same, and chances of success in each location also remain the same, what is the probability of finding the cat in time now? [4 marks]

(c) Whilst in the kitchen you hear a clear meow and you are certain she is in there, but you only have 10 minutes before your mum returns. The time to find the cat in the kitchen, $X$ minutes, has probability density function

$$f(x) = \begin{cases} 0, & x < 2; \\ 4/(x+2)^2\,, & x \geq 2. \end{cases}$$

(i) Find the cumulative distribution function of $X$; [3 marks]

(ii) and the probability that you find the cat in the kitchen before your mother returns. [2 marks]

**10**. You visit a fair at which there are many games of chance to enjoy and analyse.

(a) You are approached by a shady character who offers you £10 if you roll a "double-six" within 20 rolls of two normal dice, but otherwise you pay them £10. You (rightly) suspect shenanigans, so perform a calculation.
Find the *fairest* number of attempts to roll a "double-six" in this game. [5 marks]

(b) In another game, you shoot paintballs at a circular target. The trajectory of the balls is somewhat unpredictable, so there is only a 60% chance that you hit the target. If you take five shots, find the probability that:

(i) all shots hit the target; [2 marks]

(ii) fewer than three shots hit the target. [3 marks]

There is a regular hexagon drawn inside the target, such that the vertices touch the edge, giving six segments around the target. Two are coloured blue and the others green. You win £10 for hitting a blue segment, £5 for hitting a green segment, £1 for hitting inside the hexagon, and nothing for a miss.

(iii) Find the expected winnings from a single shot. [6 marks]

(c) You can try plate spinning. The time you can keep the plate spinning, $X$ minutes, has probability density function

$$f(x) = \begin{cases} 0, & x < 0; \\ 3/(x+1)^4\,, & x \geq 0. \end{cases}$$

Find the probability that you can keep the plate spinning for more than one minute. [4 marks]

**11**. A Spanish treasure galleon has to pass through pirate infested seas on its journey from the Americas to Spain. The chance of being attacked and captured by pirates is equally likely on every voyage, independent of any previous ones. The number of voyages, $N$, that a galleon makes before being captured by pirates has probability mass function (PMF):

$$p(n) = \left(\frac{13}{15}\right)^n \frac{2}{13}, \quad \text{for } n = 1, 2, 3, \ldots \text{ (and zero otherwise.)}$$

(You may assume that the galleon's voyages will be completed successfully unless attacked by pirates.)

(a) What type of probability distribution does $N$ follow? [1 mark]

(b) Verify that $p(n)$ is a valid PMF. [4 marks]

(c) Write down (or derive) the cumulative distribution function of $N$. [2 marks]

(d) Find the probability that the galleon:

(i) is captured on the third voyage; [2 marks]

(ii) makes fewer than 4 voyages before being captured; [2 marks]

(iii) survives more than 20 voyages before being captured; [2 marks]

(iv) survives at least 5 more voyages, given that she has made 15 already. [3 marks]

(e) Calculate from first principles the expected number of voyages made before the galleon is captured. [4 marks]

HINT: You may quote for $|x| < 1$:

$$1 + x + x^2 + x^3 + \cdots = (1 - x)^{-1}, \text{ and}$$
$$1 + 2x + 3x^2 + 4x^3 + \cdots = (1 - x)^{-2}$$

**12**. (a) Let

$$A = \{n \in \mathbb{N} : n \leq 4\}; \quad B = \{1, 2, 5, 6\}; \quad C = \{1, 2, 7, 8\}$$

be possible events of rolling a fair 8-sided die.

(i) Show that $A, B, C$ are *pairwise* independent. [4 marks]

(ii) Are they *completely* independent? [2 marks]

Let $D = \{2n - 1 : n \leq 4, n \in \mathbb{N}\}$.

(iii) Show that $A, B, D$ are *completely* independent. [4 marks]

(b) Let $X$ have the following cumulative distribution function

$$F(x) = \begin{cases} 0, & x < 0; \\ \frac{1}{32}(6x^2 - x^3), & 0 \leq x \leq 4; \\ 1, & x > 4 \end{cases}$$

(i) Find and sketch the corresponding probability density function of $X$. [5 marks]

(ii) Hence find the mean of $X$, [2 marks]

(iii) and the variance of $X$. [3 marks]

**13**. A point $Q$ is chosen at random on a line $AB$ of length $5$, such that each point on the line is equally likely. Let $X$ be the length of $AQ$.

(a) Write down the cumulative distribution function (CDF) $F_X$ of $X$. [2 marks]

(b) A second random point is chosen on the line. Let $Y$ be the distance of the furthest of the two points from $A$. Find the CDF of $Y$. [3 marks]

(c) For $n$ random points on the line, let $Z$ be the distance of the furthest of all the points from $A$. Show that the CDF of the random variable $Z$ is:

$$F_Z(x) = \begin{cases} 0 & x < 0 \\ x^n/5^n & 0 \leq x \leq 5 \\ 1 & x > 5. \end{cases}$$

[3 marks]

(d) Find the median of $Z$ in terms of $n$. [2 marks]

(e) Find the probability density function of $Z$, and hence show that:

$$\mathbb{E}(Z^m) = \frac{5^m n}{m + n}, \quad \text{for } m, n = 1, 2, 3, \ldots.$$

[6 marks]

(f) Hence, find the mean and variance of $Y$. [4 marks]