PRIFYSGOL ABERYSTWYTH UNIVERSITY

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 1 / SEMESTER 1 EXAMINATIONS

IONAWR / JANUARY 2026

MA10510 – Algebra

The questions on this paper are written in English.

Amser a ganiateir - 2 awr

- Rhoddir marciau llawn am atebion cyflawn i bob cwestiwn yn Rhan A ac i dri cwestiwn yn Rhan B.
- Yn Rhan B, rhoddir credyd am y tri ateb gorau.
- Ni cheir defnyddio cyfrifianellau.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

Time allowed - 2 hours

- Full marks will be given for complete answers to all questions in Section A and to three questions in Section B.
- In Section B, credit will be given for the best three answers.
- Calculators are not permitted.
- Students may submit answers to this paper in either Welsh or English.

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Section A

1. Consider the following sets:

$$A = \{a \in \mathbb{N} : a(a^2-2)(a^2-4) = 0\},$$
$$B = \{b \in \mathbb{Z} : |b| \leq 3\},$$
$$C = \{1, 3, 7\}.$$

Specify the following explicitly:

a) A; b) $A \cup B$; c) $A \times C$; d) $|\mathcal{P}(C)|$.

[5 marks]

2. Find rational numbers q, a and r such that

$$0.3\overline{081} = q + \sum_{k=0}^{\infty} ar^k.$$

Hence, express $0.3\overline{081}$ as a fraction in its lowest terms. [5 marks]

3. Consider the complex numbers $a = 8 - 4i$ and $b = -1 + i$. Determine the real and imaginary parts of the following complex numbers:

a) $z_1 = \frac{1}{2}a + 3b$; b) $z_2 = ab$; c) $z_3 = e^{\frac{3\pi}{2}i}$; d) $z_4 = a^{-1}$.

[5 marks]

4. Determine all $x \in \mathbb{R}$ satisfying $\dfrac{2}{x-1} > 4 - x$. [5 marks]

5. Prove by mathematical induction that $2n^3 + 3n^2 + n$ is divisible by 6 for every integer $n \geq 1$. [5 marks]

6. Resolve $\dfrac{8-9x}{3x^2-10x+8}$ into partial fractions. [5 marks]

7. Consider the polynomial $p(x) = 3x^3 - 13x^2 + kx + 65$, where k is some fixed real number. Given that $3 - 2i$ is a root of $p(x)$, find all roots of $p(x)$ and the value of k. [5 marks]

8. Exactly one of the following functions is a bijection:

$$f : [0, \infty) \to [-1, \infty),\ f(x) = x^3 + x^2 + 2;$$
$$g : [5, \infty) \to [2, \infty),\ g(x) = x^2 - 10x + 27.$$

a) State which one is **not** a bijection and justify why it is not a bijection.

b) Specify the inverse of the bijection fully.

[5 marks]

Section B

9. Consider the polynomials

$$a(x) = 2x^5 + 11x^4 + 11x^3 - 13x^2 - 35x - 56 \text{ and } b(x) = (x^2 - x - 2)(x^2 + 6x + 9).$$

a) There exists a polynomial $q(x)$ such that

$$a(x) = q(x)b(x) + r(x),$$

where $r(x) = 4x^3 + 28x^2 + 22x - 38$. Determine $q(x)$. [5 marks]

b) Hence, resolve $\dfrac{a(x)}{b(x)}$ into partial fractions. [15 marks]

10. a) Solve the following equations for $z \in \mathbb{C}$. Write each solution in exponential form, in terms of its principal argument, explicitly.

(i) $z^6 + 4\sqrt{2} + 4\sqrt{2}i = 0$. [6 marks]

(ii) $z^6 - 8z^3 + 64 = 0$. [8 marks]

b) Use De Moivre's Theorem to show that $\sin\frac{3\pi}{5}$ is a root of the polynomial $p(x) = 16x^4 - 20x^2 + 5$. [6 marks]

11. a) Let a and b be non-negative real numbers. Prove that

$$a < b \iff a^2 < b^2.$$

[4 marks]

b) Determine all $x \in \mathbb{R}$ satisfying

$$\left|2x^2 - 3x - 2\right| < x^2 + 2.$$

[6 marks]

c) Determine all $x \in \mathbb{R}$ satisfying

$$\left|\frac{2x+1}{x+2}\right| < \left|\frac{x+3}{x+6}\right|.$$

[10 marks]

12. a) Let $T_0 := 0$ and for each integer $n \geq 1$, let $T_n := \sum_{r=1}^{n} r$.

(i) State a formula for T_n in terms of n only, when $n \geq 1$. [2 marks]

(ii) Prove that $T_n^2 - T_{n-1}^2 = n^3$ for every integer $n \geq 1$. [4 marks]

(iii) Hence, or otherwise, prove that $\sum_{r=1}^{n} r^3 = T_n^2$ for every integer $n \geq 1$. [4 marks]

b) Define a sequence (u_n) of positive integers by $u_1 = 1$, $u_2 = 2$ and

$$u_{n+1} = 2u_n + 2u_{n-1} \text{ for all } n > 1.$$

Prove by strong mathematical induction that $u_n = \frac{1}{2\sqrt{3}}(\alpha^n - \beta^n)$ for all positive integers n, where α and β are the two distinct solutions of the equation $x^2 - 2x - 2 = 0$ and $\alpha > 0$. [*Hint: calculate α and β first.*] [10 marks]

13. a) State the binomial theorem for positive integer exponents. [4 marks]

b) State the general term in the binomial expansion of $\left(3x^3 - \dfrac{1}{3x^4}\right)^{15}$ in as simple form as possible. Hence determine the coefficient of x^{-25} stating your answer as a fraction in its lowest terms. [6 marks]

c) Expand the first 4 terms of the binomial $(1+x)^{\frac{1}{3}}$, where $|x| < 1$, simplifying the coefficients as far as possible. Use your result to determine the decimal expansion of $\sqrt[3]{8.72}$ rounded to 4 decimal places. Show all your workings clearly to gain full credit. [10 marks]