PRIFYSGOL ABERYSTWYTH UNIVERSITY

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 1/ SEMESTER 1 EXAMINATIONS

IONAWR / JANUARY 2026

MA20110 - Real Analysis

The questions on this paper are written in English.

Amser a ganiateir - 2 awr

- Gellir rhoi cynnig ar bob cwestiwn.
- Rhoddir mwy o ystyriaeth i berfformiad yn rhan B wrth bennu marc dosbarth cyntaf.
- Cyfrifianellau Casio FX-83 neu FX-85 YN UNIG a ganiateir.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

Time allowed - 2 hours

- All questions may be attempted.
- Performance in section B will be given greater consideration in assigning a first class mark.
- Casio FX-83 or FX-85 calculators ONLY may be used.
- Students may submit answers to this paper in either Welsh or English.

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Section A

1. Define $f : [-\pi, \pi) \to \mathbb{R}$ by

$$f(x) = \begin{cases} -x^2 & \text{if } -\pi \leq x < 0, \\ x^2 & \text{if } 0 \leq x < \pi. \end{cases}$$

(a) Sketch the 2π-periodic extension of f on the interval $[-3\pi, 3\pi]$. [2 marks]

(b) Show that the Fourier series of f is given by

$$\sum_{n=1}^{\infty} \left(\frac{2\pi(-1)^{n+1}}{n} + \frac{4((-1)^n - 1)}{\pi n^3} \right) \sin nx.$$

[8 marks]

(c) State the Fourier series of $g : [-\pi, \pi) \to \mathbb{R}$ defined by $g(x) = f(x) + \cos 7x$. [2 marks]

2. By considering the sequence of partial sums, show that the infinite series

$$\sum_{k=1}^{\infty} \frac{1}{(k+5)(k+6)}$$

is convergent and determine its value. [5 marks]

3. Determine the convergence or otherwise of the following series, stating clearly which convergence test you use and your reasoning:

$$\text{(a)} \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+4}}; \quad \text{(b)} \sum_{n=1}^{\infty} \frac{2n+3}{n^2+n-1}; \quad \text{(c)} \sum_{n=1}^{\infty} \frac{(n+1)^2}{2n^2+7n+3}.$$

[4,5,4 marks]

4. For each of the following power series, determine all values of x for which the series converges. Find the radius of convergence in each case.

$$\text{(a)} \sum_{k=0}^{\infty} \frac{3^k}{(k+2)^2}(x-1)^k; \quad \text{(b)} \sum_{k=0}^{\infty} \frac{(k+1)!}{\exp(k^2)} x^k.$$

[6,5 marks]

5. Find the Taylor series of $\cos 2x$ about $x = 0$. Prove that the series converges to $\cos 2x$ for $x > 0$. [5,5 marks]

6. For $n \in \mathbb{N}$, define $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 2 - 6nx & \text{if } 0 \leq x < 1/(3n), \\ 0 & \text{if } 1/(3n) \leq x \leq 1 - 1/(3n), \\ 6n(x-1) + 2 & \text{if } 1 - 1/(3n) \leq x \leq 1. \end{cases}$$

(a) Determine the pointwise limit of (f_n). [4 marks]

(b) Show that (f_n) does not converge uniformly to its pointwise limit. [3 marks]

7. (a) Explain what is meant by saying that a function $f : \mathbb{R}^n \to \mathbb{R}$ is *continuous at* **a** for some $\mathbf{a} \in \mathbb{R}^n$. [2 marks]

(b) Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} (3xy^4 + 8xy)/(|x| + y^4) & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0), \end{cases}$$

and

$$g(x,y) = \begin{cases} 4xy/(y + x^2) & \text{if } y \neq -x^2, \\ 0 & \text{if } y = -x^2. \end{cases}$$

Prove that f is continuous at $(0,0)$ but that g is not continuous at $(0,0)$. [5,5 marks]

SECTION B BEGINS ON THE NEXT PAGE

Section B

8. Let $f : \mathbb{R} \to \mathbb{R}$ be an even 2π-periodic function that satisfies $f(x) = \pi - x$ on the interval $[0, \pi]$.

(a) Show that the Fourier series of f is given by

$$\frac{\pi}{2} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}.$$

[8 marks]

(b) Use the Fourier Convergence Theorem to deduce that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

[5 marks]

9. Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are real series, satisfying $b_n \neq 0$ for each $n \in \mathbb{N}$, and that

$$\lim_{n\to\infty} \left| \frac{a_n}{b_n} \right| = l \text{ for some } l > 0.$$

Prove that if $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then $\sum_{n=1}^{\infty} b_n$ converges. [7 marks]

10. (a) State and prove the *Integral Test* for real series. [2,9 marks]

(b) Determine the convergence or otherwise of the series

$$\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}.$$

[6 marks]

11. Define

$$\mathcal{L} = \{f : [1,4] \to \mathbb{R} : |f(x) - f(y)| \leq 3|x - y| \text{ for every } x, y \in [1,4]\}.$$

Let $(f_n) \subset \mathcal{L}$ be such that f_n converges uniformly to f for some $f : [1,4] \to \mathbb{R}$. Define $g : [1,4] \to \mathbb{R}$ by $g(x) = x^2 + f(x)$. Demonstrate there exists $M > 0$ such that $|g(x) - g(y)| \leq M|x - y|$ for every $x, y \in [1,4]$. [7 marks]

12. Let $h : \mathbb{R}^2 \to \mathbb{R}$ be defined by $h(x, y) = 2xy^2$. Working directly with the definition of derivative, prove that h is differentiable everywhere in $\mathbb{R}^2$. [6 marks]

END OF EXAM PAPER