PRIFYSGOL ABERYSTWYTH UNIVERSITY

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 1 / SEMESTER 1 EXAMINATIONS

IONAWR / JANUARY 2026

MA26010 – Distributions and Estimation

The questions on this paper are written in English.

Amser a ganiateir - 2 awr

- Gellir rhoi cynnig ar bob cwestiwn.
- Rhoddir mwy o ystyriaeth i berfformiad yn Rhan B wrth bennu marc dosbarth cyntaf.
- Cyfrifianellau Casio FX-83 neu FX-85 YN UNIG a ganiateir.
- Darperir tablau ystadegol.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

Time allowed - 2 hours

- All questions may be attempted.
- Performance in Section B will be given greater consideration in assigning a first class mark.
- Casio FX-83 or FX-85 calculators ONLY may be used.
- Statistical tables will be provided.
- Students may submit answers to this paper in either Welsh or English.

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Section A

1. The five observations 21, 26, 12, 6, 14 are a random sample from the uniform distribution over the interval $(0, \theta)$ where $\theta > 0$ is an unknown parameter. Justify why 31.6, 32, and 31.2 might all be considered unbiased estimates of θ. [6 marks]

2. A random variable X has probability density function (pdf)

$$f_X(x) = 2(1-x), \quad \text{for } 0 < x < 1 \text{ (and zero elsewhere).}$$

Consider the transformation $Y = 2X/(1-X)$.

(a) Sketch the graph of the transformation, and hence deduce the range of Y. Indicate the ranges of X and Y in the sketch. [3 marks]

(b) Is the transformation one-to-one? [1 mark]

(c) Find the pdf of Y. [6 marks]

3. A random variable X has pdf

$$f_X(x) = \frac{2}{9}x, \quad \text{for } 0 < x < 3 \text{ (and zero elsewhere).}$$

Consider the transformation $Y = (X-1)^2$.

(a) Sketch the graph of the transformation, and hence deduce the range of Y. Indicate the ranges of X and Y in the sketch. [3 marks]

(b) Is the transformation one-to-one? [1 mark]

(c) Find the pdf of Y. [6 marks]

4. The discrete random variables X and Y have the joint probability distribution given by the following table:

		x		
		0	**1**	**2**
y	**0**	$5a$	$3a$	a
	1	$2a$	$4a$	$6a$

(a) Find the value of a. [1 mark]

(b) Find the marginal probability mass functions (pmf) of X and Y. [5 marks]

(c) Calculate the covariance between X and Y. [3 marks]

(d) Are X and Y independent? Justify your answer. [2 marks]

(e) Calculate $\mathbb{E}[\cos(\pi X)Y]$. [2 marks]

5. The moment generating function (mgf) of a random variable X is given by

$$M_X(t) = \frac{1}{3}e^{-5t} + \frac{k}{1-3t}, \quad t < \frac{1}{3}.$$

(a) Find the values of (i) k, (ii) $\mathbb{E}[e^{X/4}]$, and (iii) $\mathbb{E}[X]$. [4 marks]

(b) Using the standard series $e^t = \sum_{n=0}^{\infty} t^n/n!$ and $(1-t)^{-1} = \sum_{n=0}^{\infty} t^n$, expand $M_X(t)$ into a power series up to 2nd order, and hence find $\mathbb{E}[X^2]$. [4 marks]

6. A random variable X has pdf

$$f_X(x) = (\ln 2)\, 2^x, \quad \text{for } 0 < x < 1 \text{ (and zero elsewhere).}$$

Find the mgf of X. [4 marks]

7. Let X and Y be *independent* random variables, and consider the variable $Z = X + Y$.

(a) *State* the formula with which $\mathbb{E}[f(X)g(Y)]$ can be calculated for any functions f and g, if $\mathbb{E}[f(X)]$ and $\mathbb{E}[g(Y)]$ are given. [1 mark]

(b) Using the formula in (a), *derive* the formula for the mgf of Z, in terms of the mgfs of X and Y. Include all relevant steps in the derivation. [3 marks]

8. The continuous random variables X and Y have the joint range

$$\{(x, y) : 0 < x < 1,\, 0 < y < 1\}$$

and pdf

$$f_{XY}(x, y) = x^2 + cy.$$

(a) Find the value of c. [2 marks]

(b) Find the marginal pdf of X. [2 marks]

(c) Find the conditional pdf of Y given that $X = \frac{1}{3}$. [2 marks]

(d) (i) Sketch the subset of the joint range consisting of those (x, y) which satisfy $2y + x < 1$. (ii) Hence calculate $\mathbb{P}(2Y + X < 1)$. [6 marks]

(e) Calculate $\mathbb{P}(Y < \frac{1}{2} | X = \frac{1}{3})$. [3 marks]

Section B

9. A random sample $X_1, \ldots, X_{120}$ is drawn from the Standard Normal distribution. Let $Q = \sum_{i=1}^{120} X_i^2$.

(a) Identify the distribution of Q as a member of a standard family and write down its mean, variance, and moment generating function. [3 marks]

(b) Find the value of c such that $\mathbb{P}(Q > c) = 0.4$. [2 marks]

(c) Find $\mathbb{E}[Q^4]$. [3 marks]

10. A random sample of six observations was drawn from the exponential distribution $\mathrm{Expon}(3\theta)$, where $\theta > 0$ is the unknown parameter. The observations turned out to be 1.6, 4.1, 2.9, 0.3, 2.4, 2.7.

(a) Show that the likelihood function is $L(\theta) = 729\,\theta^6\, e^{-42\theta}$. [4 marks]

(b) Find the maximum likelihood estimate of θ. [5 marks]

11. The six observations $3,\ 1,\ 11,\ 9,\ -1,\ 6$ are a random sample from the uniform distribution over the interval $(5 - 3\theta, 5 + 3\theta)$ where $\theta > 0$ is an unknown parameter.

(a) (i) Explain why the sample mean does not yield any information on θ. [1 mark]

(ii) Show that $\mathbb{E}[X^2] = 25 + 3\theta^2$ for a single sample point X. [3 marks]

(iii) Hence find the moment estimate for θ. [3 marks]

(b) Find an unbiased estimate for θ based on the smallest value in the sample. [4 marks]

12. The random variables X and Y have their joint pdf given by

$$f_{XY}(x, y) = 4y^2 e^{-x-2y}, \quad x > 0,\ y > 0$$

(and is zero otherwise). Let $U = \frac{1}{5}Y$ and $V = X + 2Y$.

(a) Show that this transformation is one-to-one. [3 marks]

(b) Calculate the Jacobian $\frac{\partial(u,v)}{\partial(x,y)}$ of the transformation. Deduce the value of $\frac{\partial(x,y)}{\partial(u,v)}$. [3 marks]

(c) Determine and sketch the joint range of (U, V). [4 marks]

(d) Deduce the joint pdf of (U, V). [4 marks]

(e) Find the pdfs of U and V, and identify them as members of the Gamma family of distributions. [8 marks]