PRIFYSGOL ABERYSTWYTH UNIVERSITY

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 1/ SEMESTER 1 EXAMINATIONS

IONAWR / JANUARY 2026

MA30210 - Norms and Differential Equations

The questions on this paper are written in English.

Amser a ganiateir - 2 awr

Time allowed - 2 hours

- Gellir rhoi cynnig ar bob cwestiwn.
- Rhoddir mwy o ystyriaeth i berfformiad yn rhan B wrth bennu marc dosbarth cyntaf.
- Cyfrifianellau Casio FX-83 neu FX-85 YN UNIG a ganiateir.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

- All questions may be attempted.
- Performance in section B will be given greater consideration in assigning a first class mark.
- Casio FX-83 or FX-85 calculators ONLY may be used.
- Students may submit answers to this paper in either Welsh or English.

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Section A

1. Determine whether the following functions $F_i : \mathbb{R}^3 \to \mathbb{R}$ define norms on $\mathbb{R}^3$. For $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$,

(a) $F_1(\mathbf{x}) = 200|x_1|$; (b) $F_2(\mathbf{x}) = |3x_1 + 2x_2 + x_3|$;
(c) $F_3(\mathbf{x}) = 6|x_1| + 7|x_2| + |x_3|$; (d) $F_4(\mathbf{x}) = |x_1| + 2x_2^2 + 4|x_3|$;
(e) $F_5(\mathbf{x}) = (|x_1|^{1/4} + |x_2|^{1/4} + |x_3|^{1/4})^4$.

[17 marks]

2. For $n \in \mathbb{N}$, define $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = 3x^{n+1}$ for $x \in [0,1]$.

Show that $f_n \to 0$ as $n \to \infty$ in $(C[0,1], \|.\|_1)$, but that (f_n) does not converge in $(C[0,1], \|.\|_{\sup})$, where

$$\|f\|_1 = \int_0^1 |f(x)|dx, \ \|f\|_{\sup} = \sup\{|f(x)| : x \in [0,1]\}.$$

[3,5 marks]

3. Let $(V, \langle ., . \rangle)$ be an inner product space.

(a) State the *Cauchy-Schwarz* inequality. [2 marks]

(b) Prove that $\|.\| : V \to \mathbb{R}$ defined by $\|x\| = \langle x, x \rangle^{1/2}$ for $x \in V$ is a norm on V. [6 marks]

(You may use the Cauchy-Schwarz inequality without proof.)

4. Let U be a real vector space with basis $\{e_1, e_2, e_3, e_4\}$. Define norms $\|.\|_w$ and $\|.\|_\infty$ on U by

$$\begin{aligned} \|x\|_w &= 3|x_1| + |x_2| + 2|x_3| + 5|x_4|; \\ \|x\|_\infty &= \max\{|x_1|, |x_2|, |x_3|, |x_4|\}; \end{aligned}$$

for $x = \sum_{i=1}^4 x_i e_i \in U$. Demonstrate that $\|.\|_w$ and $\|.\|_\infty$ are equivalent norms on U. [5 marks]

(You may use any characterisation of equivalent norms you know without proof.)

5. For $n \in \mathbb{N}$, define $g_n : [0,1] \to \mathbb{R}$ by

$$g_n(x) = \begin{cases} 2nx & \text{if } 0 \le x < 1/(2n), \\ 1 & \text{if } 1/(2n) \le x \le 1 - 1/(2n), \\ 2n(1-x) & \text{if } 1 - 1/(2n) < x \le 1. \end{cases}$$

Prove that (g_n) is a Cauchy sequence in $(C[0,1], \|.\|_1)$, but not in $(C[0,1], \|.\|_{\sup})$, where $\|.\|_1$ and $\|.\|_{\sup}$ are defined as in question 2. [4,4 marks]

6. Suppose that $(x_n), (y_n)$ are Cauchy sequences in a normed vector space $(V, \|.\|)$. Prove that $(\|2x_n + 3y_n\|)$ is a convergent sequence in $(\mathbb{R}, |.|)$. [7 marks]

(You may assume $(\mathbb{R}, |.|)$ is complete without proof.)

7. Define $e : C[0,2] \to \mathbb{R}$ by $e(f) = 2f(0) - f(2)$.

(a) Demonstrate that e is linear. [2 marks]

(b) Prove that $e : (C[0,2], \|.\|_{\sup}) \to (\mathbb{R}, |.|)$ is bounded, but that $e : (C[0,2], \|.\|_1) \to (\mathbb{R}, |.|)$ is not bounded. [2,5 marks]

8. Let $(X, \|.\|)$ be a Banach space.

(a) For $D \subset X$, explain what is meant by saying that $T : D \to D$ is a *contraction mapping*. [2 marks]

(b) Let $T : X \to X$ be such that $T^3 : X \to X$ is a contraction mapping, where $T^3(x) = T(T(T(x)))$ for $x \in X$. Prove that T has a unique fixed point. [6 marks]

SECTION B BEGINS ON THE NEXT PAGE

Section B

9. Let U be a real vector space with basis $\{e_1, e_2, e_3\}$. Define, for $x = \sum_{i=1}^{3} x_i e_i \in U$ and $\mu \in \mathbb{R}$,

$$F(x) = \{x_1^2 + \mu x_1 x_3 + x_2^2 + x_3^2\}^{1/2}.$$

(a) Prove that F defines a norm on U if and only if $|\mu| < 2$. [10 marks]

(b) For $|\mu| < 2$, is (U, F) a Banach space? Stating any result you need without proof, give a brief reason. [2 marks]

10. Let $(X, \|.\|)$ be a real normed vector space, and let $L : X \to \mathbb{R}$ be a linear mapping. Prove that L is continuous if and only if Ker$(L) = \{x \in X : L(x) = 0\}$ is closed. [10 marks]

(You may use, without proof, any equivalent characterisations of continuous linear mappings you know.)

11. Define $T : (C[0,1], \|.\|_{\sup}) \to (C[0,1], \|.\|_{\sup})$ by

$$T(f)(x) = \int_0^x t^6 f(t) dt$$

for $f \in C[0,1]$, $x \in [0,1]$.

(a) Explain why $T(f) \in C[0,1]$ for $f \in C[0,1]$. [2 marks]

(b) Prove that T is linear and bounded and that $\|T\| = 1/7$, where $\|T\|$ denotes the operator norm of T. [8 marks]

(c) For $\Psi \in C[0,1]$, how many functions $f \in C[0,1]$ satisfy the integral equation

$$f(x) = \int_0^x t^6 f(t) dt + \Psi(x) \text{ for every } x \in [0,1]?$$

Justify your answer. [6 marks]

12. State and prove *Banach's Contraction Mapping Theorem*. [2,10 marks]

END OF EXAM PAPER