PRIFYSGOL ABERYSTWYTH UNIVERSITY

**ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS**

**ARHOLIADAU SEMESTER 1 / SEMESTER 1 EXAMINATIONS**

**IONAWR / JANUARY 2026**

# MA30510 - Differential Geometry of Curves and Surfaces

The questions on this paper are written in English.

**Amser a ganiateir - 2 awr**

**Time allowed - 2 hours**

- Gellir rhoi cynnig ar bob cwestiwn.
- Rhoddir mwy o ystyriaeth i berfformiad yn rhan B wrth bennu marc dosbarth cyntaf.
- Cyfrifianellau Casio FX83 neu FX85 YN UNIG a ganiateir.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

- All questions may be attempted.
- Performance in Section B will be given greater consideration in assigning a first-class mark.
- Casio FX83 or FX85 calculators ONLY may be used.
- Students may submit answers to this paper in either Welsh or English.

You may require the following integral:

$$\int \sqrt{x^2-1}\,dx \;=\; \frac{1}{2}\Big(x\sqrt{x^2-1} \;-\; \cosh^{-1}(x)\Big) + c.$$

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## Section A

**1**. (a) Find a parametrisation for the level curve

$$\frac{x^2}{4} + y^2 = c,$$

where $(x, y) \in \mathbb{R}^2$ and $c \in \mathbb{R}$. What is the domain of your parametrisation? [4 marks]

(b) Using your answer from part (a), sketch the level curves for $c = 1,\ 4,\ 9$. In your sketch, label each ellipse with its corresponding value of $c$. [4 marks]

**2**. Show that the length $S$ of the curve

$$y(x) = 2x^{3/2},$$

over the interval $0 \leq x \leq 1$ is

$$S = \frac{2}{27}\left(10\sqrt{10} - 1\right).$$

[8 marks]

**3**. (a) Provide the definition of a *regular curve* and a *unit speed curve*. [4 marks]

(b) Consider the cycloid $\boldsymbol{\alpha}(t) = (x(t), y(t)) = (t - \sin(t), 1 - \cos(t))$.

(i) Determine whether $\boldsymbol{\alpha}(t)$ is a regular curve for all values of t. If not, state the values of t for which it is not regular. You may find the following identity helpful

$$\sin^2\frac{\theta}{2} = \frac{1 - \cos(\theta)}{2},$$

[7 marks]

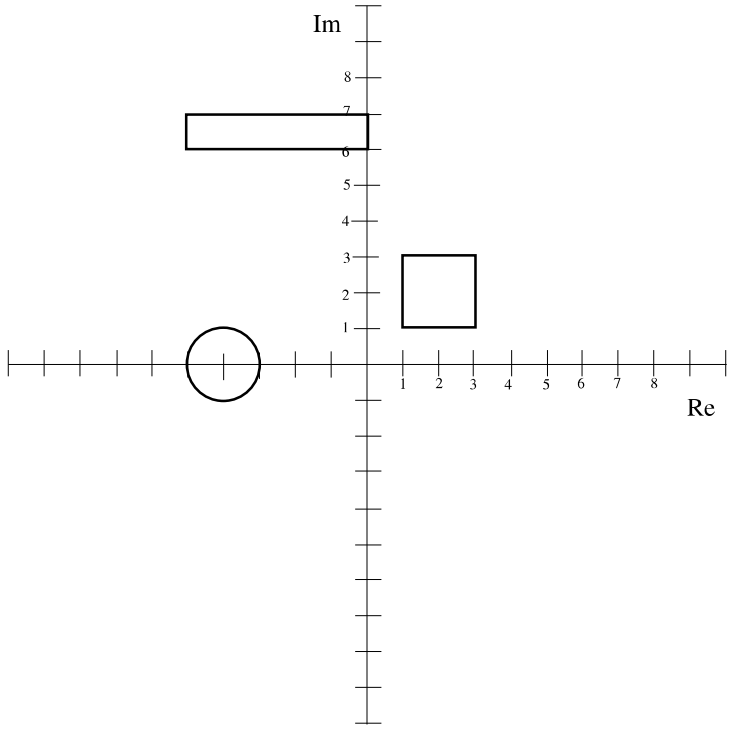
(ii) Find a unit speed parametrization of $\boldsymbol{\alpha}(t)$. [5 marks]

(iii) Sketch the curve $\boldsymbol{\alpha}(t)$ on $x$-$y$ axes for the range $0 \leq t \leq 2\pi$ and indicate the value of $\boldsymbol{\alpha}(t)$ at key points. [4 marks]

**4**. The diagram below shows a square, a rectangle, and a circle in the $z$-plane. Sketch the images of these objects under the following transformations. For the circle, determine its curvature both before and after each mapping.

(a) $w = z - (1 + i)$, [4 marks]

(b) $w = \frac{1}{2}e^{-i\pi}z$, [4 marks]




**5**. Consider the curve $y(x) = \cosh(x)$.

(a) Show that the curvature of $y(x)$ is given by

$$\kappa(x) = \operatorname{sech}^2 x,$$

[6 marks]

(b) For what value(s) of $x$ is the curvature a maximum? What is the maximum curvature? [4 marks]

(c) At $x = 0$, find the equation of the *osculating circle*, i.e., the circle that is tangent to the curve $y(x)$ at that point and has the same curvature. [3 marks]

**6**. Explain what is meant by the *osculating plane*, the *normal plane*, and the *rectifying plane* of a space curve with unit tangent vector $\boldsymbol{T}$, normal vector $\boldsymbol{N}$ and binormal vector $\boldsymbol{B}$. [6 marks]

**7**. Compute the first fundamental form of the surface

$$\mathbf{S}(u, v) = \left(u,\ v,\ u^2 + v^2\right), \quad u, v \in \mathbb{R}.$$

[7 marks]

## Section B

**8**. Two curves are called *Bertrand mates* if they share the same principal normal vector at corresponding points *(i.e. there is a one-to-one matching between points on the two curves such that the line segment joining the matched points is parallel to this common normal)*.

Let $\boldsymbol{\alpha}(s)$ be a unit–speed space curve with Frenet frame $(\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s))$, with smooth curvature $\kappa(s)$ and torsion $\tau(s)$. Suppose $\boldsymbol{\alpha}$ admits a Bertrand mate $\boldsymbol{\beta}(s)$.

(a) Explain why the relation between $\boldsymbol{\alpha}(s)$ and $\boldsymbol{\beta}(s)$ must be of the form

$$\boldsymbol{\beta}(s) = \boldsymbol{\alpha}(s) + \lambda(s)\,\mathbf{N}(s),$$

where $\lambda(s)$ is a smooth function. [2 marks]

(b) Differentiate this expression with respect to $s$ and show that

$$\boldsymbol{\beta}'(s) = \big(1 - \lambda(s)\,\kappa(s)\big)\,\mathbf{T}(s) + \lambda'(s)\,\mathbf{N}(s) + \lambda(s)\,\tau(s)\,\mathbf{B}(s).$$

[3 marks]

(c) Using the fact that $\boldsymbol{\beta}(s)$ also has $\mathbf{N}(s)$ as its principal normal, show that

$$\boldsymbol{\beta}'(s) = \big(1 - a\,\kappa(s)\big)\,\mathbf{T}(s) + a\,\tau(s)\,\mathbf{B}(s), \tag{A}$$

where $a$ is a constant. [5 marks]

(d) Let $\tilde{s}$ denote the arc-length parameter along $\boldsymbol{\beta}(s)$ and define the unit tangent of $\boldsymbol{\beta}(s)$ at the matched point by

$$\tilde{\mathbf{T}}(s) = \frac{\boldsymbol{\beta}'(s)}{\big|\boldsymbol{\beta}'(s)\big|}. \tag{B}$$

Let $\phi(s)$ be the angle between $\mathbf{T}(s)$ and $\tilde{\mathbf{T}}(s)$, so that

$$\mathbf{T}(s) \cdot \tilde{\mathbf{T}}(s) = \cos\phi(s).$$

Assume that, for a Bertrand mate,

$$\tilde{\mathbf{T}}'(s) = \tilde{\kappa}(s)\,\frac{d\tilde{s}}{ds}\,\mathbf{N}(s),$$

(where $\tilde{\kappa}(s)$ is a scalar function of $s$) hence show that $\phi(s)$ is constant. [4 marks]

(e) Recall from (B) that $\tilde{\mathbf{T}}(s)$ is the unit tangent vector to $\boldsymbol{\beta}(s)$, i.e. it points in the same direction as $\boldsymbol{\beta}'(s)$. Using equation (A) for $\boldsymbol{\beta}'(s)$ together with the fact that the angle $\phi$ between $\mathbf{T}(s)$ and $\tilde{\mathbf{T}}(s)$ is constant, show that there exist constants $a, b$ such that

$$a\,\kappa(s) + b\,\tau(s) = 1$$

for all $s$. [4 marks]

**9.** (a) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (u(x, y), v(x, y))$ with $u, v \in C^1$, and suppose the Cauchy–Riemann equations hold:

$$u_x = v_y, \qquad u_y = -\,v_x.$$

(i) Show that, under these conditions, the coefficients of the first fundamental form are

$$E = u_x^2 + v_x^2, \qquad G = E, \qquad F = 0.$$

[6 marks]

(ii) Deduce that the first fundamental form can be written in the form

$$I = \lambda^2\,(dx^2 + dy^2),$$

and prove that $\lambda = |f'(z)|$. [3 marks]

(b) Let $f(x)$ be a smooth function and consider the surface of revolution

$$\mathbf{S}(u, v) = \big(u\cos v,\ u\sin v,\ f(u)\big),$$

obtained by rotating the curve $z = f(x)$ in the $x$–$z$ plane about the $z$-axis.

(i) Compute the coefficients of the first fundamental form of $\mathbf{S}(u, v)$. State the first fundamental form. [6 marks]

(ii) Recall that a parametrisation is conformal if its first fundamental form is a scalar multiple of $du^2 + dv^2$. Show that $\mathbf{S}(u, v)$ is conformal to the Euclidean plane if and only if

$$f(u) = \pm\frac{1}{2}\left(u\sqrt{u^2 - 1}\ -\ \cosh^{-1}(u)\right) + c,$$

where $c$ is a constant. [4 marks]

**10**. Let $S \subset \mathbb{R}^3$ be a regular surface with smooth patch $\boldsymbol{\sigma}(u, v)$ and unit normal $\mathbf{n}$. Denote the first and second fundamental forms by $I$ and $II$. Let $\boldsymbol{\alpha}(s) = \boldsymbol{\sigma}(u(s), v(s))$ be a *unit-speed* curve on $S$.

(a) Let $\boldsymbol{\sigma}_u = \partial\boldsymbol{\sigma}/\partial u$ and $\boldsymbol{\sigma}_v = \partial\boldsymbol{\sigma}/\partial v$. Show that the tangent vector to the curve is given by,

$$\mathbf{T} = \boldsymbol{\alpha}'(s) = u'(s)\,\boldsymbol{\sigma}_u + v'(s)\,\boldsymbol{\sigma}_v,$$

and state why $I(\mathbf{T}, \mathbf{T}) = \mathbf{T} \cdot \mathbf{T} = 1$. [2 marks]

(b) Demonstrate that the normal curvature is given by

$$k_n \;= \boldsymbol{\alpha}''(s) \cdot \mathbf{n}(s) \;= \frac{II(\mathbf{T}, \mathbf{T})}{I(\mathbf{T}, \mathbf{T})},$$

where $\mathbf{n}(s)$ is the unit surface normal. [9 marks]

(c) Let $k(s)$ be the curvature of the space curve $\boldsymbol{\alpha}(s)$ and let $\mathbf{N}(s)$ denote its unit principal normal from the Frenet–Serret frame. If $\phi$ is the angle between $\mathbf{N}$ and the surface normal $\mathbf{n}$ at $\boldsymbol{\alpha}(s)$, deduce *Meusnier's Theorem*:

$$\boxed{k_n \;=\; k\,\cos\phi\,.}$$

[2 marks]