PRIFYSGOL ABERYSTWYTH UNIVERSITY

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 1 / SEMESTER 1 EXAMINATIONS

IONAWR / JANUARY 2026

MA32410 – Graphs and Networks

The questions on this paper are written in English.

Amser a ganiateir - 2 awr

- Gellir rhoi cynnig ar bob cwestiwn.
- Rhoddir mwy o ystyriaeth i berfformiad yn Rhan B wrth bennu marc dosbarth cyntaf.
- Cyfrifianellau Casio FX-83 neu FX-85 YN UNIG a ganiateir.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

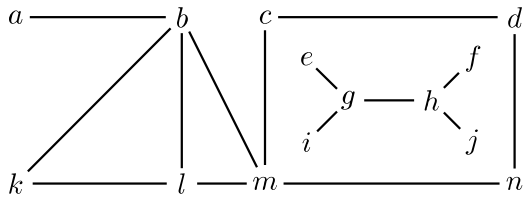
Time allowed - 2 hours

- All questions may be attempted.
- Performance in Section B will be given greater consideration in assigning a first class mark.
- Casio FX-83 or FX-85 calculators ONLY may be used.
- Students may submit answers to this paper in either Welsh or English.

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Section A

1. Give an example of each of the following in the graph below:



(a) a closed walk of length at least 2 that is not a trivial closed walk;

(b) a circuit that is not a cycle;

(c) a component containing vertex h;

(d) two bridges.

[8 marks]

2. For each of the following cases, determine whether or not there exists a graph G, with vertex set V and edge list E, satisfying the stated properties. Justify your answers either by providing an example of a graph satisfying the stated properties, or by giving a brief explanation of why such a graph cannot exist.

(a) $|E| = 13$ and G is regular of degree 5.

(b) $|V| = 10$ and G is regular of degree 3.

(c) $|E| = 6$ and G is bipartite and regular of degree 2.

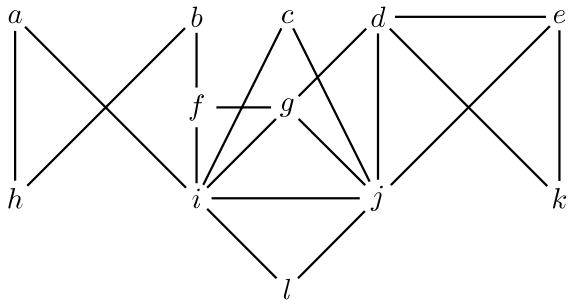
(d) $|V| = 3$, $|E| = 4$ and G is simple.

[8 marks]

3. Prove that in a graph with exactly n vertices, if there is a walk from a vertex u to a vertex v, with $u \neq v$, then there is a path from u to v, and the length of the path is at most $n - 1$. [5 marks]

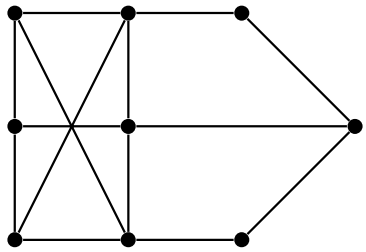
4. (a) State the definitions of an Eulerian circuit and an Eulerian trail in terms of walks in a graph. [4 marks]

(b) Determine whether or not the graph below contains an Eulerian circuit or an Eulerian trail. Justify your answer. If it does, write down an Eulerian circuit or an Eulerian trail.



[5 marks]

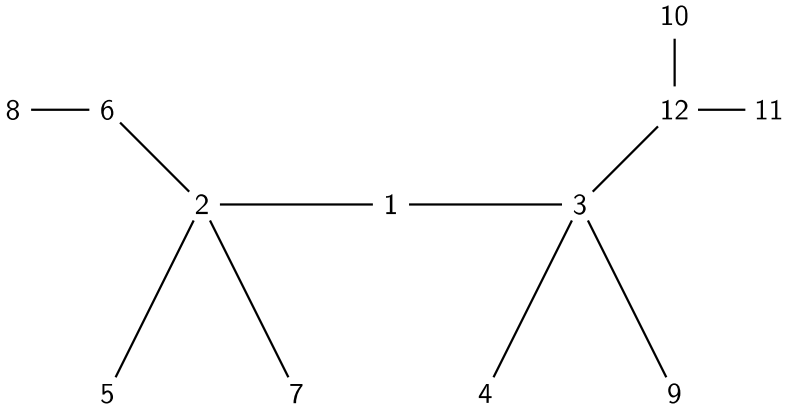
5. Consider the graph below.



(a) Show that the graph is bipartite by labelling two parts of its vertex set. [3 marks]

(b) Determine whether the graph is Hamiltonian. Justify your answer. [3 marks]

6. (a) Determine the Prüfer code of the labelled tree



[4 marks]

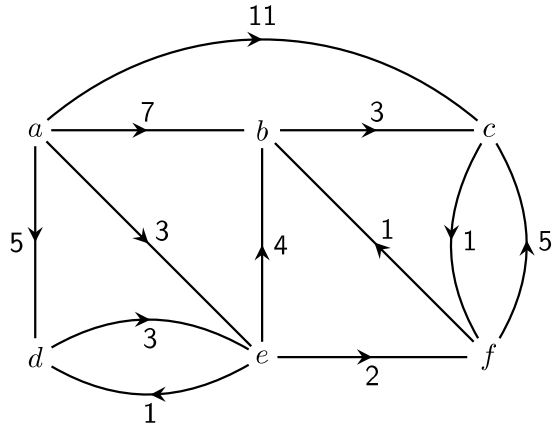
(b) Determine the edges of the labelled tree, with vertices labelled $1, 2, \ldots, 11$, that has Prüfer code $11, 1, 10, 11, 1, 11, 2, 1, 10$. [4 marks]

7. By applying Prim's Algorithm or Kruskal's Algorithm, determine the edges and weight of a minimum weight spanning tree in the graph whose weight matrix is

$$\begin{pmatrix} - & 2 & - & - & - & 3 & - & - & - \\ 2 & - & 1 & 1 & - & - & - & - & - \\ - & 1 & - & 1 & 4 & - & - & - & - \\ - & 1 & 1 & - & 4 & 2 & 4 & - & - \\ - & - & 4 & 4 & - & - & 2 & 3 & - \\ 3 & - & - & 2 & - & - & - & - & 4 \\ - & - & - & 4 & 2 & - & - & 2 & 1 \\ - & - & - & - & 3 & - & 2 & - & 2 \\ - & - & - & - & - & 4 & 1 & 2 & - \end{pmatrix}.$$

[8 marks]

8. Apply Dijkstra's algorithm to the weighted digraph below taking a as the origin.



Using the algorithm, determine the shortest distance from a to any other vertex and draw a shortest path tree. [8 marks]

9. Apply the Warshall-Floyd algorithm to the weighted digraph with weight matrix

$$\begin{pmatrix} 0 & \infty & 4 & 8 & 4 \\ 7 & 0 & 5 & \infty & \infty \\ \infty & \infty & 0 & \infty & 6 \\ 4 & 3 & \infty & 0 & \infty \\ 3 & 3 & \infty & 1 & 0 \end{pmatrix}.$$

Give the final optimum policy matrix. Use the final optimum policy matrix to find the shortest path from vertex 2 to vertex 4. [10 marks]

Section B

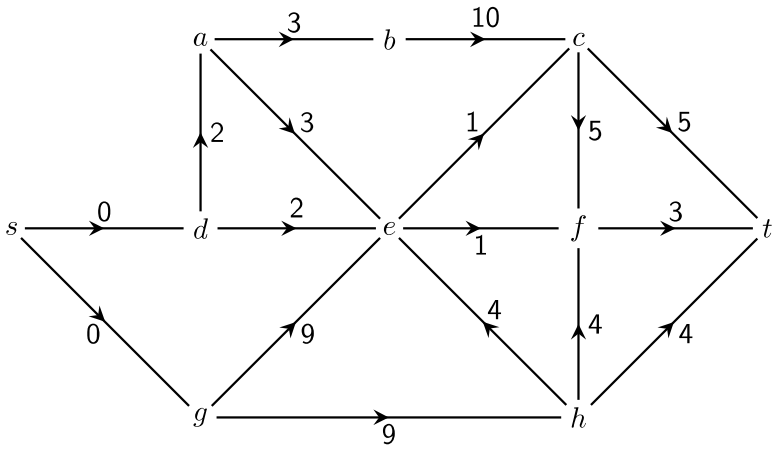
10. State and prove Euler's Theorem for graphs.

You may assume that for a graph in which every vertex has even degree, every edge is on a circuit. [12 marks]

11. (a) Prove that every non-trivial tree has at least two leaves. [4 marks]

(b) Let T be a labelled tree with exactly $n \geq 2$ vertices. Prove by induction on n that the label of a vertex of degree $d \geq 1$ in T appears exactly $d-1$ times in the Prüfer code of T. [6 marks]

12. The weighted digraph below represents the dependencies and durations of activities a, b, c, d, e, f, g, h of a project.



(a) Apply the longest path algorithm to the digraph to find:

(i) the longest distance to each vertex from vertex s; and

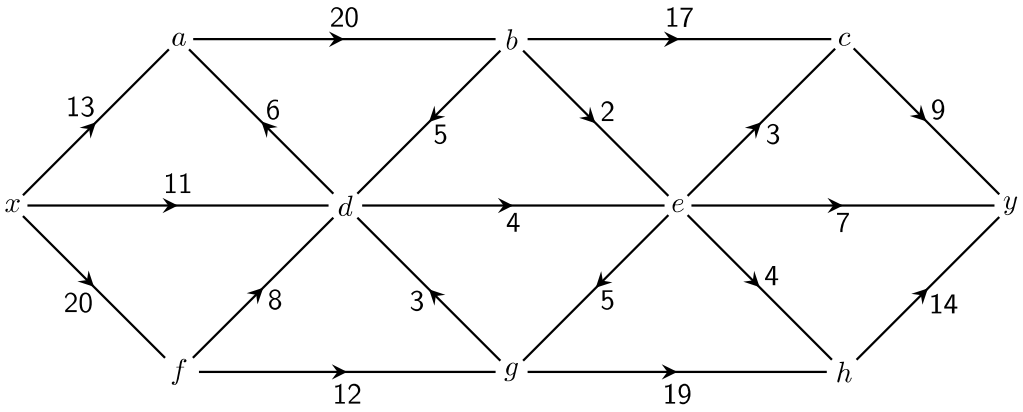
(ii) the longest distance from each vertex to vertex t.

[8 marks]

(b) Hence, find the earliest completion time of the project, a critical path, and the earliest and latest start times of each activity. [4 marks]

13. (a) Explain briefly why in any transport network if the value of some flow is equal to the capacity of some cut, then the flow is a maximum flow and the cut is a minimum cut. [4 marks]

(b) Apply the Ford-Fulkerson algorithm to the following transport network. Show the final label on each vertex, flow along each edge and the minimum cut. State the maximum flow value.



[12 marks]