PRIFYSGOL ABERYSTWYTH UNIVERSITY

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 1 / SEMESTER 1 EXAMINATIONS

IONAWR / JANUARY 2026

MA32610 – Topology

The questions on this paper are written in English.

Amser a ganiateir - 2 awr **Time allowed - 2 hours**

- Gellir rhoi cynnig ar bob cwestiwn.
- Rhoddir mwy o ystyriaeth i berfformiad yn rhan B wrth bennu marc dosbarth cyntaf.
- Cyfrifianellau Casio FX-83 neu FX-85 YN UNIG a ganiateir.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

- All questions may be attempted.
- Performance in section B will be given greater consideration in assigning a first class mark.
- Casio FX-83 or FX-85 calculators ONLY may be used.
- Students may submit answers to this paper in either Welsh or English.

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Section A

1. (a) State the definition of an open set and of a closed set in a metric space (X, d).[3]

(b) Consider the following subsets in $\mathbb{R}$ (with the usual Euclidean metric):

$$\{2, 3\}, \quad (-1, 1), \quad (-1, 0) \cup (0, 1), \quad (-1, 0) \cup \{1\}, \quad \mathbb{R}, \quad \mathbb{Q}$$

Classify these sets as 'open and not closed' or 'closed and not open' or 'open and closed' or 'not open and not closed'.
Choose one of the cases where you decided that it is closed and give a detailed argument why it is closed. [10]

(c) Prove, starting from your definition in (a), that the union of finitely many closed sets in a metric space is closed. [5]

(d) Give an example showing that the union of infinitely many closed sets in a metric space is not always closed. No proof required here. [2]

[20 marks]

2. (a) What does it mean for metrics d and d' to be Lipschitz-equivalent? [2]

(b) Give an example of two metrics on the interval $[0, 1]$ which are not Lipschitz-equivalent and give a proof that they are not. [5]

[7 marks]

3. (a) State the axioms of a topological space. [3]

(b) If X and Y are topological spaces, what does it mean for $f : X \to Y$ to be continuous? [2]

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 1$ if $x < 0$ and $f(x) = 0$ if $x \geq 0$. Prove with the definition you provided in (b) that f is not continuous (for the usual topology on $\mathbb{R}$). [3]

[8 marks]

4. (a) For a subset A of a topological space X state the definitions of the interior $\mathring{A}$ and the closure $\bar{A}$ of A. [3]

(b) Prove from the definition that the interior $\mathring{A}$ is always an open set. [3]

(c) Prove from the definition that the closure $\bar{A}$ is always a closed set. [4]

[10 marks]

5. (a) What is a homeomorphism? [2]

(b) Write down a homeomorphism from the circle given by the equation $x^2 + y^2 = 1$ to the ellipse given by the equation $2x^2 + 3y^2 = 1$ (both considered as subspaces of $\mathbb{R}^2$ with the usual topology). No proof required. [3]

[5 marks]

6. (a) State the definition of connectedness for a topological space. [2]

(b) Suppose X and Y are topological spaces which are homeomorphic. Prove that if X is connected then so is Y. [4]

(c) Give a detailed argument why the letters C and D (both considered as subspaces of $\mathbb{R}^2$ with the usual topology) are not homeomorphic. [5]

[11 marks]

7. (a) State the definition of an open cover and of a compact subset in a topological space. [3]

(b) Prove from the definitions that a convergent sequence together with its limit point is a compact set. [6]

[9 marks]

Section B

8. Let (X, d) be a metric space and $A \subset X$ a subspace. Prove that the closure $\bar{A}$ of A (according to the definition of closure in a topological space) is equal to the set of all limit points of sequences which are contained in A and convergent in X. [12 marks]

9. We consider $\mathbb{Q}$, the set of rational numbers, as a subset of the set of real numbers $\mathbb{R}$ (with its usual topology).

(a) Show that $\mathbb{Q}$ is dense in $\mathbb{R}$. [3]

(b) Let Y be any topological space. Show that if f and g are continuous functions from $\mathbb{R}$ to Y whose restrictions to $\mathbb{Q}$ coincide then $f = g$. [3]

We now equip $\mathbb{Q}$ with the subspace topology.

(c) What are the open subsets of $\mathbb{Q}$ in this topology? [2]

(d) Is $\mathbb{Q}$ discrete (in the sense that for each rational number there is an open set which contains only this number)? [3]

(e) Is $\mathbb{Q}$ Hausdorff? [3]

(f) Is $\mathbb{Q}$ connected? [4]

(g) Is $\mathbb{Q}$ compact? [3]

Justify your answers. [21 marks]

10. Consider the sphere $S^2 := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ (with the subspace topology from $\mathbb{R}^3$) and the equivalence relation $(x, y, z) \sim (x', y', z')$ if and only if $x = x', y = y', z = z'$ or $x = -x', y = -y', z = -z'$ (antipodal point).

(a) State the definition of the quotient space and of the quotient topology. [3]

(b) Is the resulting quotient space $S^2/\sim$ compact? [6]

(c) Is the resulting quotient space $S^2/\sim$ Hausdorff? [8]

Give full proofs for your answers in (b) and (c), explicitly quoting all the results (as known from lectures) that you use in your proof.

[17 marks]

END OF PAPER