PRIFYSGOL ABERYSTWYTH UNIVERSITY

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 1 / SEMESTER 1 EXAMINATIONS

IONAWR / JANUARY 2026

MA33410 – Theory of Elasticity

The questions on this paper are written in English.

Amser a ganiateir - 2 awr

- Gellir rhoi cynnig ar bob cwestiwn.
- Rhoddir mwy o ystyriaeth i berfformiad yn Rhan B wrth bennu marc dosbarth cyntaf.
- Cyfrifianellau Casio FX-83 neu FX-85 YN UNIG a ganiateir.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

Time allowed - 2 hours

- All questions may be attempted.
- Performance in Section B will be given greater consideration in assigning a first class mark.
- Casio FX-83 or FX-85 calculators ONLY may be used.
- Students may submit answers to this paper in either Welsh or English.

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Supplementary material

The strain - displacement and rotation - displacement relations for a symmetric infinitesimal strain tensor ε_{ij}, rotation tensor Ω_{ij}, and displacement u_i on and within an elastic body take the form

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \quad \Omega_{ij} = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right).$$

The Saint-Venant compatibility equations for a symmetric infinitesimal strain tensor ε_{ij}, in the coordinate system (x, y, z), take the form

$$\frac{\partial^2\varepsilon_{xx}}{\partial y^2} + \frac{\partial^2\varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2\varepsilon_{xy}}{\partial x\partial y}, \qquad \frac{\partial^2\varepsilon_{yy}}{\partial z^2} + \frac{\partial^2\varepsilon_{zz}}{\partial y^2} = 2\frac{\partial^2\varepsilon_{yz}}{\partial y\partial z},$$

$$\frac{\partial^2\varepsilon_{zz}}{\partial x^2} + \frac{\partial^2\varepsilon_{xx}}{\partial z^2} = 2\frac{\partial^2\varepsilon_{xz}}{\partial x\partial z}, \qquad \frac{\partial^2\varepsilon_{xx}}{\partial y\partial z} = \frac{\partial}{\partial x}\left(\frac{\partial\varepsilon_{xz}}{\partial y} + \frac{\partial\varepsilon_{xy}}{\partial z} - \frac{\partial\varepsilon_{yz}}{\partial x}\right),$$

$$\frac{\partial^2\varepsilon_{yy}}{\partial x\partial z} = \frac{\partial}{\partial y}\left(-\frac{\partial\varepsilon_{xz}}{\partial y} + \frac{\partial\varepsilon_{xy}}{\partial z} + \frac{\partial\varepsilon_{yz}}{\partial x}\right), \quad \frac{\partial^2\varepsilon_{zz}}{\partial x\partial y} = \frac{\partial}{\partial z}\left(\frac{\partial\varepsilon_{xz}}{\partial y} - \frac{\partial\varepsilon_{xy}}{\partial z} + \frac{\partial\varepsilon_{yz}}{\partial x}\right).$$

Hooke's law relating the stress field σ_{ij} and the strain field ε_{ij} within an elastic, homogeneous and isotropic material takes the form

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}, \quad \varepsilon_{ij} = \frac{1}{2\mu}\left(\sigma_{ij} - \frac{\lambda}{3\lambda + 2\mu}\sigma_{kk}\delta_{ij}\right),$$

where λ, μ are Lamé coefficients.

Cauchy's equations for a body at equilibrium state that the stress field σ_{ij} and the body forces F_i experienced by an elastic body must be related as

$$\frac{\partial\sigma_{ij}}{\partial x_j} + F_i = 0.$$

Differential operators: In spherical coordinates (r, θ, ϕ) acting on the orthonormal basis $(\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi)$, we have:

$$\text{grad}(f) = \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{e}_\theta + \frac{1}{r\sin(\theta)}\frac{\partial f}{\partial\phi}\hat{e}_\phi,$$

$$\text{div}(\mathbf{u}) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right) + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)u_\theta\right) + \frac{1}{r\sin(\theta)}\frac{\partial u_\phi}{\partial\phi},$$

$$\text{curl}(\mathbf{u}) = \frac{1}{r\sin(\theta)}\left[\frac{\partial}{\partial\theta}\left(\sin(\theta)u_\phi\right) - \frac{\partial u_\theta}{\partial\phi}\right]\hat{e}_r + \frac{1}{r}\left[\frac{1}{\sin(\theta)}\frac{\partial u_r}{\partial\phi} - \frac{\partial}{\partial r}\left(ru_\phi\right)\right]\hat{e}_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(ru_\theta\right) - \frac{\partial u_r}{\partial\theta}\right]\hat{e}_\phi.$$

The **Airy stress function** ϕ in polar coordinates (r, θ) is a scalar function such that

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}, \quad \sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2}, \quad \sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right).$$

In polar coordinates (r, θ), the **biharmonic equation** for a function $\phi(r, \theta)$ takes the form

$$\nabla^4\phi = \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]^2\phi = 0.$$

Section A

1. State which of the following equations are valid expressions in indicial notation. If they are not valid, state why not.

(a)
$$A_i = B_{ik}C_j.$$
[2 marks]

(b)
$$A_{ij} = B_i C_{jk} D_k.$$
[2 marks]

(c)
$$A_{i,j} = B_k C_i D_{kj} F_k.$$
[2 marks]

(d)
$$A_{ij,jj} = B_i C_{jj} D_{k,k} + F_{ij} G_j.$$
[2 marks]

(e)
$$A_{ij,kl} = B_{ik,jl} + F_{kl,is} G_{s,j}.$$
[2 marks]

2. Rewrite the following expressions in the form of a matrix equation, assuming that all rank one tensors are represented as column vectors.

(a)
$$A_{ij} = B_{kj} C_{si} D_s F_k.$$
[4 marks]

(b)
$$A_i = B_{ij} C_k F_{kj} + F_k G_{jj} D_{ik}.$$
[4 marks]

3. Consider a homogeneous, isotropic, elastic cube defined within the region $x, y, z \in [0, 1]$. The displacement field within and on the cube is given by

$$\mathbf{u} = \begin{bmatrix} s(2x^2 + 3ys) \\ s(y^2 + z - 3xs) \\ s(y - z^2) \end{bmatrix},$$

where $0 < s \ll 1$ is a small constant. The equations needed to answer this question are provided in the supplementary material.

(a) Determine the infinitesimal strain field $\boldsymbol{\varepsilon}$ and rotation tensor $\boldsymbol{\Omega}$ within the elastic body. [10 marks]

(b) Without evaluating them, state whether this infinitesimal strain field $\boldsymbol{\varepsilon}$ will satisfy the Saint-Venant compatibility equations. Justify your answer. [3 marks]

(c) State whether an infinitesimal strain approximation is suitable in this case. Explain your reasoning. [2 marks]

4. A unit cube of jelly sits on a flat plate. It is subject to uniform atmospheric pressure p on its faces. The cube is pulled down by gravity with the medium experiencing force per unit volume $F_z = -\rho g$, where g is the gravitational constant and ρ is the density of the medium. No other forces are acting on or within the jelly. The system is assumed to be at equilibrium. The jelly is approximated as a homogeneous, isotropic, elastic body.

All equations needed to answer this question are given in the supplementary material. Part (b) can be answered without completing part (a).

(a) (i) State the body forces $\boldsymbol{F}$ acting within the cube. [2 marks]

(ii) Assume that the cube is in its principal coordinate system. Use Cauchy's equations to demonstrate that the stress tensor in the cube is in the form

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx}(y,z) & 0 & 0 \\ 0 & \sigma_{yy}(x,z) & 0 \\ 0 & 0 & \rho g z + f(x,y) \end{bmatrix},$$

for some unknown function f. [6 marks]

(b) (i) State the normal vectors and tractions on the following faces of the cube

- $\{x = 1;\ y, z \in [0,1]\}$;
- $\{y = 1;\ x, z \in [0,1]\}$;
- $\{z = 1;\ x, y \in [0,1]\}$.

[3 marks]

(ii) Hence, determine the unique stress tensor for the jelly cube. [6 marks]

5. The Cauchy stress tensor at a point in a homogeneous, isotropic, elastic medium is given by

$$\boldsymbol{\sigma} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}.$$

(a) Show that the principal values are $\lambda_1 = 4$, $\lambda_2 = 1$, $\lambda_3 = -2$. Determine the associated principal axes, $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, $\boldsymbol{n}_3$, for the stress $\boldsymbol{\sigma}$ at this point. [10 marks]

(b) State the stress tensor at this point when transformed to the principal coordinate system. [2 marks]

(c) Show that the infinitesimal strain tensor $\boldsymbol{\varepsilon}$ experienced at this point in the elastic medium, in its principal coordinate system, is given by

$$\boldsymbol{\varepsilon} = \frac{1}{2\mu} \begin{bmatrix} 4 - \dfrac{3\lambda}{3\lambda + 2\mu} & 0 & 0 \\ 0 & 1 - \dfrac{3\lambda}{3\lambda + 2\mu} & 0 \\ 0 & 0 & -2 - \dfrac{3\lambda}{3\lambda + 2\mu} \end{bmatrix},$$

where $\lambda, \mu > 0$ are the Lamé coefficients for the medium. [4 marks]

(d) Compute an estimate for the volume change being experienced at this point in the medium. State whether, on average, the material is undergoing extension or compression at this point. Justify your answer. [4 marks]

Section B

6. The bathysphere of a submersible designed to reach the bottom of the Mariana Trench can be approximated as a hollow, homogeneous and isotropic elastic sphere. The sphere has inner radius r_0 and outer radius r_1. It is subject to uniform internal pressure p_0, and uniform external pressure p_1. We assume that the sphere is at equilibrium, and consider the system in spherical coordinates (r, θ, ϕ). The Navier-Lamé equations for the bathysphere are given by

$$2(1-\nu)\text{grad}\,(\text{div}\,(\boldsymbol{u})) - (1-2\nu)\text{curl}\,(\text{curl}\,(\boldsymbol{u})) = 0,$$

where ν is the Poisson's ratio. The form of the differential operators is provided in the supplementary material. Parts (c)-(d) can be completed independently of (a)-(b).

(a) Explain how the displacement $\boldsymbol{u}$, and its radial component u_r, depend on the angles (θ, ϕ). Hence, simplify the Navier-Lamé equations for this case. [6 marks]

(b) Solve the resulting equation to demonstrate that the radial displacement can be represented in the form

$$u_r = Ar + \frac{B}{r^2},$$

for some constants A, B. [2 marks]

(c) The stress-displacement relations for the sphere are given by

$$\sigma_{rr} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\frac{du_r}{dr} + \frac{2\nu}{r}u_r\right], \quad \sigma_{\theta\theta} = \sigma_{\phi\phi} = \frac{E}{(1+\nu)(1-2\nu)}\left[\frac{1}{r}u_r + \nu\frac{du_r}{dr}\right],$$

where E is the Young's modulus.

Formulate appropriate boundary conditions for the stress on the surfaces of the hollow sphere. Hence, determine the constants A, B and demonstrate that the stress experienced within the medium is given by

$$\sigma_{rr} = \frac{1}{r_1^3 - r_0^3}\left[\frac{r_0^3 r_1^3(p_0+p_1)}{r^3} - (p_1 r_1^3 + p_0 r_0^3)\right],$$

$$\sigma_{\theta\theta} = \sigma_{\phi\phi} = -\frac{1}{r_1^3 - r_0^3}\left[\frac{r_0^3 r_1^3(p_0+p_1)}{2r^3} + (p_1 r_1^3 + p_0 r_0^3)\right].$$

[12 marks]

(d) The yield stress criterion provides a condition which, when satisfied, indicates that the medium will exceed the yield point of its constituent material. For the hollow sphere with material dependent yield stress Y it can be formulated as

$$|\sigma_{\theta\theta} - \sigma_{rr}| = Y.$$

Assume that the bathysphere has inner radius $r_0 = 1$ m, and internal pressure identical to that at the surface $p_0 = 10^5$ Pa. The sphere is comprised of titanium alloy, which has a yield strength of $Y = 8.3 \times 10^8$ Pa. Determine the minimum outer radius r_1 needed to prevent the sphere exceeding the yield point when diving to the bottom of the Mariana Trench, where the external pressure is 10^8 Pa. Provide your answer in the form of a cube root. [10 marks]

7. A homogeneous, isotropic, elastic half-space is subject to vertical loading of a magnitude P at a single point on the surface. The resulting stress field can be generated in polar coordinates (r, θ) from the Airy stress function

$$\phi(r, \theta) = -\frac{P}{\pi} r\theta \sin(\theta).$$

The equations needed to solve this question are provided in the supplementary material. Note that all of parts (a)-(d) can be completed independently of one another.

(a) Demonstrate that ϕ is a valid Airy stress function. [5 marks]

(b) Demonstrate that the stress field induced by the point loading has components

$$\sigma_{rr} = -\frac{2P\cos(\theta)}{\pi r}, \quad \sigma_{\theta\theta} = \sigma_{r\theta} = 0.$$

[4 marks]

(c) For a plane strain system in polar coordinates, the radial and hoop stress, infinitesimal strain and displacement are related as

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} = \frac{1+\nu}{E}\left[(1-\nu)\sigma_{rr} - \nu\sigma_{\theta\theta}\right],$$

$$\varepsilon_{\theta\theta} = \frac{1}{r}\left[\frac{\partial u_\theta}{\partial \theta} + u_r\right] = \frac{1+\nu}{E}\left[(1-\nu)\sigma_{\theta\theta} - \nu\sigma_{rr}\right],$$

where E is the Young's modulus, and ν the Poisson's ratio. Demonstrate that the displacement field generated by the point loading takes the form

$$u_r = -\frac{2P(1-\nu^2)}{\pi E}\cos(\theta)\log(r) + f(\theta),$$

$$u_\theta = \frac{2P(1+\nu)}{\pi E}\sin(\theta)\left[\nu + (1-\nu)\log(r)\right] - \int f(\theta)\, d\theta + g(r),$$

for some unknown functions $f(\theta)$ and $g(r)$. [5 marks]

(d) For a plane strain system in polar coordinates, the shearing strains $\varepsilon_{r\theta}$ are related to the stress and displacement as

$$\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{1}{r}u_\theta\right) = \frac{1+\nu}{E}\sigma_{r\theta}.$$

Determine the equations $f(\theta)$ and $g(r)$ must satisfy. Hence, show that taking

$$f(\theta) = -\frac{P(1+\nu)(1-2\nu)}{\pi E}\theta\sin(\theta), \quad g(r) = 0,$$

provides a valid unique displacement in the half-plane (in the absence of arbitrary rigid displacements). [6 marks]