PRIFYSGOL ABERYSTWYTH UNIVERSITY

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 1 / SEMESTER 1 EXAMINATIONS

IONAWR / JANUARY 2026

MA34110 – Partial Differential Equations

The questions on this paper are written in English.

Amser a ganiateir - 2 awr

- Gellir rhoi cynnig ar bob cwestiwn.
- Rhoddir mwy o ystyriaeth i berfformiad yn Rhan B wrth bennu marc dosbarth cyntaf.
- Cyfrifianellau Casio FX-83 neu FX-85 YN UNIG a ganiateir.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

Time allowed - 2 hours

- All questions may be attempted.
- Performance in Section B will be given greater consideration in assigning a first class mark.
- Casio FX-83 or FX-85 calculators ONLY may be used.
- Students may submit answers to this paper in either Welsh or English.

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Section A

1. A partial differential equation can be written in the form

$$Lu = g,$$

where L is an operator, u the dependent variable, and g a known function of the independent variables.

(a) What conditions must L and g satisfy for the PDE to be *linear inhomogeneous*? [2 marks]

(b) For each of the following equations, state its order and classify it as nonlinear, linear inhomogeneous, or linear homogeneous:

(i) $u_{xx} + xu_x - u = \sin x$;
(ii) $u_t + uu_x = 0$;
(iii) $e^y u_{yy} + 3u_y = y^2$;
(iv) $(u_x)^3 + u_{xy} = xy$. [8 marks]

(c) Classify the following second–order PDEs as parabolic, elliptic, or hyperbolic:

(i) $u_{tt} - 4u_{tx} + 4u_{xx} = 0$;
(ii) $2u_{xx} + 5u_{xy} + 2u_{yy} + u = 0$. [4 marks]

(d) Find the general solution $u(x, y)$ of the PDE

$$u_{xx} = 6x + e^y.$$

[3 marks]

2. Using the method of characteristics and clearly explaining each step, solve the following boundary-value problems:

(a) $\begin{cases} 4u_x + 9u_y = 0, \\ u(0, y) = \sin(4y); \end{cases}$ (b) $\begin{cases} e^{2x}u_x + \frac{1}{2}u_y = 0, \\ u(0, y) = \cos(4y). \end{cases}$

[6,8 marks]

3. Suppose the temperature $u = u(x, t)$ of a thin rod satisfies the heat equation

$$u_t = u_{xx},$$

for $(x, t) \in (0, 2) \times (0, 5)$, with the following initial and boundary conditions:

$$\begin{cases} u(x, 0) = \frac{1}{2}x(2 - x), & x \in [0, 2]; \\ u(0, t) = \sin\left(\frac{\pi}{5}t\right), & t \in [0, 5]; \\ u(2, t) = \frac{t}{5}, & t \in [0, 5]. \end{cases}$$

(a) Specifically for this domain, state the definition of the *parabolic boundary* Π. [3 marks]

(b) State the maximum and minimum principles for the heat equation. [2 marks]

(c) Without solving the PDE (and clearly naming any theorems you invoke), determine the minimum and maximum temperatures attained on $(0, 2) \times (0, 5)$. [8 marks]

(d) State all pairs of (x, t) values for which the temperature is maximised. [2 marks]

4. Consider the following one–dimensional wave equation problem, describing the displacement u of a string of length L with fixed ends, whose initial displacement and velocity are given by known functions f and g:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & (x,t) \in (0,L) \times (0,T); \\ u(0,t) = u(L,t) = 0, & t \in [0,T]; \\ u(x,0) = f(x), \quad u_t(x,0) = g(x), & x \in [0,L]. \end{cases}$$

(a) Using the method of separation of variables, find the general solution of this problem in the form

$$u(x,t) = \sum_{k=1}^{\infty} X_k(x) T_k(t),$$

You are not required to evaluate any Fourier coefficients. [11 marks]

(b) Hence, or otherwise, solve the problem for $c = 1$, $L = T = \pi$,

$$f(x) = 5\sin(3x), \qquad g(x) = 4\sin(3x).$$

[3 marks]

5. Clearly stating any results you use, solve the following Cauchy problem for the inhomogeneous wave equation:

$$\begin{cases} u_{tt} - u_{xx} = 6, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = \sin x, & x \in \mathbb{R}, \\ u_t(x,0) = 2x, & x \in \mathbb{R}. \end{cases}$$

You are not required to simplify trigonometric expressions but should otherwise simplify fully. [10 marks]

SECTION B BEGINS ON THE NEXT PAGE

Section B

6. Giving your answer in as simple a form as possible, use the method of characteristics to solve the PDE

$$2xu_x + 3yu_y = 5u,$$

subject to the condition $u(1, y) = y^3$. [10 marks]

7. Consider the following initial–boundary value problem, which models small transverse vibrations of a slender, homogeneous beam of length L with hinged ends:

$$\begin{cases} u_{tt} + u_{xxxx} = 0, & 0 < x < L,\ t > 0, \\ u(0,t) = u(L,t) = 0, \quad u_{xx}(0,t) = u_{xx}(L,t) = 0, & t \geq 0, \\ u(x,0) = 0, \quad u_t(x,0) = 0, & 0 \leq x \leq L. \end{cases}$$

Define the energy

$$E(t) \;=\; \tfrac{1}{2}\int_0^L \left[u_t^2 + u_{xx}^2\right] \mathrm{d}x.$$

(a) Show by differentiating under the integral and integrating by parts (twice) that

$$\frac{\mathrm{d}E}{\mathrm{d}t} \equiv 0.$$

[9 marks]

(b) Hence, clearly stating which boundary or initial conditions inform your conclusions, find the value of $E(t)$. What can be deduced regarding u_t and u_{xx}? [5 marks]

(c) Explain why the above (together with boundary conditions) force $u(x,t) \equiv 0$. [4 marks]

8. Let u be a solution of the Cauchy problem for the one-dimensional convection–diffusion equation:

$$\begin{cases} u_t + au_x - Du_{xx} = 0, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}, \end{cases} \tag{1}$$

where $a \in \mathbb{R}$ and $D > 0$ are constants, and define the Fourier transform of u with respect to x as

$$\mathcal{F}[u] = \bar{u}(\xi, t) = \int_{-\infty}^{\infty} u(x,t)e^{i\xi x}\mathrm{d}x.$$

(a) Taking the Fourier transform with respect to x, and clearly stating any properties that you use, show that the Fourier transform $\bar{u}(\xi, t)$ satisfies

$$\frac{\partial \bar{u}}{\partial t} - \left(ia\xi + D\xi^2\right)\bar{u}(\xi, t) = 0.$$

[4 marks]

(b) Hence solve this equation to find an explicit expression for $\bar{u}(\xi, t)$ in terms of $\bar{f}(\xi)$. [5 marks]

9. Consider the following second order linear PDE with variable coefficients:

$$x^2 u_{xx} - 2xyu_{xy} + y^2 u_{yy} + xu_x + yu_y = 0, \qquad x > 0. \tag{2}$$

(a) Classify the PDE as elliptic, hyperbolic or parabolic. [2 marks]

(b) Using the change of variables $\xi = x$ and $\eta = xy$ or otherwise, find the general solution to (2). [11 marks]

End of exam paper