PRIFYSGOL ABERYSTWYTH UNIVERSITY

**ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS**

**ARHOLIADAU SEMESTER 1 / SEMESTER 1 EXAMINATIONS**

**IONAWR / JANUARY 2026**

# MA34610 – Hydrodynamics II

The questions on this paper are written in English.

**Amser a ganiateir - 2 awr**

- Gellir rhoi cynnig ar bob cwestiwn.
- Rhoddir mwy o ystyriaeth i berfformiad yn Rhan B wrth bennu marc dosbarth cyntaf.
- Cyfrifianellau Casio FX-83 neu FX-85 YN UNIG a ganiateir.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

**Time allowed - 2 hours**

- All questions may be attempted.
- Performance in Section B will be given greater consideration in assigning a first class mark.
- Casio FX-83 or FX-85 calculators ONLY may be used.
- Students may submit answers to this paper in either Welsh or English.

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**Useful formulae**
In the usual notation:

$$\nabla\phi = \frac{1}{h_1}\frac{\partial\phi}{\partial x_1}\underline{e}_1 + \frac{1}{h_2}\frac{\partial\phi}{\partial x_2}\underline{e}_2 + \frac{1}{h_3}\frac{\partial\phi}{\partial x_3}\underline{e}_3$$

$$\nabla.\underline{u} = \frac{1}{h_1h_2h_3}\left(\frac{\partial(h_2h_3u_1)}{\partial x_1} + \frac{\partial(h_1h_3u_2)}{\partial x_2} + \frac{\partial(h_1h_2u_3)}{\partial x_3}\right)$$

$$\nabla\wedge\underline{u} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\underline{e}_1 & h_2\underline{e}_2 & h_3\underline{e}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ h_1u_1 & h_2u_2 & h_3u_3 \end{vmatrix}$$

$$\nabla^2\phi = \frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial x_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\phi}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial x_3}\right)\right)$$

**Complex potentials for hydrodynamic structures**

Uniform stream with speed $U$ parallel to the real axis: $w(z) = Uz$.

Source of strength $m$: $w(z) = \dfrac{m}{2\pi}\mathrm{Log}(z)$.

Line vortex with circulation $\Gamma$: $w(z) = -\dfrac{i\Gamma}{2\pi}\mathrm{Log}(z)$.

Doublet / dipole with strength $\mu$: $w(z) = -\dfrac{\mu}{z}$.

## Section A

**1**. Consider the flow with velocity field $\underline{u} = Ay\underline{e}_x - Ax\underline{e}_y$, where $A$ is a constant.

(a) Determine the vorticity of this flow. [4 marks]

(b) Show that this flow is possible for an incompressible fluid. [2 marks]

(c) Determine a stream function $\psi$ for this flow. [4 marks]

(d) Sketch the streamlines and indicate on them the direction of flow. [3 marks]

(e) Explain why a complex potential does not exist for this flow. [1 mark]

**2**. Consider the flow with velocity field $\underline{u} = U\underline{e}_y$, where $U$ is a positive real constant.

(a) Describe this flow in words. [2 marks]

(b) Derive the complex potential $w(z)$ for this flow, with $z = x + iy$. [2 marks]

(c) Hence, or otherwise, find the velocity potential $\phi$ and stream function $\psi$ for this flow. [2 marks]

(d) Show that the velocity potential and stream function satisfy the Cauchy-Riemann equations. [3 marks]

(e) Re-write the velocity field as components $(u_r, u_\theta)$ in plane polar coordinates $(r, \theta)$. [3 marks]

**3**. Consider a source of strength $\Gamma$ at $z = a$, where $a$ is a real positive constant, situated close to a hard wall at $\mathrm{Re}\{z\} = 0$.

(a) Sketch an appropriate image system and use it to find the complex potential $w(z)$ for this flow. [5 marks]

(b) Give a (complex) expression for the velocity components $u - iv$ for this flow. [2 marks]

(c) Find the velocity potential $\phi$ and the stream function $\psi$ for this flow. [5 marks]

**4**. Using polar coordinates, determine whether or not the function $w(z) = z^3$ is analytic, and hence whether or not it could represent the complex potential for a fluid flow. [7 marks]

**5.** (a) State the Milne-Thomson Circle Theorem for a flow with complex potential $w = F(z)$ and a circle of radius $a$. [4 marks]

(b) A flow has complex potential $F(z) = z^2$. Use the Milne-Thomson Circle Theorem to determine the complex potential for the corresponding flow past a cylinder of radius $a$. [4 marks]

(c) Find the stream function for this flow, and explain why $|z| = a$ is a streamline. [3 marks]

**6.** Give the stream function for the flow with complex potential $w(z) = Az^{8/5}$. Hence find the velocity components and sketch the streamlines. [7 marks]

**7.** The complex potential for flow with speed $U$ and circulation $\Gamma$ at an angle $\alpha$ to the positive real axis around an ellipse in a complex $\zeta$ plane is

$$w(z) = U\left(z(\zeta)\mathrm{e}^{-i\alpha} + \frac{a^2}{z(\zeta)}\mathrm{e}^{i\alpha}\right) - \frac{i\Gamma}{2\pi}\mathrm{Log}(z(\zeta)),$$

where $z(\zeta)$ is the inverse of the conformal map $\zeta = z + \dfrac{c^2}{z}$, with $0 < c < a$.

(a) Give an expression for the velocity field in the $\zeta$ plane, $u_\zeta - iv_\zeta$, in terms of $z$. [3 marks]

(b) Determine the value of the circulation $\Gamma$ for which the velocity is zero at the trailing edge of the ellipse (the Kutta condition). [4 marks]

## Section B

**8**. (a) State Blasius theorem for the force on an object with boundary $C$ exerted by an ideal fluid with density $\rho$ in a steady flow with complex potential $w(z)$. [2 marks]

(b) Describe the flow with complex potential

$$w(z) = -Uz + \frac{m}{2\pi}\mathrm{Log}(z-k) - \frac{m}{2\pi}\mathrm{Log}(z+k),$$

where $m$ and $k$ are positive real constants. Give a rough sketch of the flow and identify a possible closed streamline. [6 marks]

(c) Use Blasius theorem to determine the force that acts on the body outlined by this streamline. [10 marks]

**9**. (a) State the Schwarz-Christoffel formula for constructing a conformal transformation from the upper half $z$ plane onto a polygonal region in a $\zeta$ plane with internal angles $\alpha_i$. [4 marks]

(b) Use the Schwarz-Christoffel formula to determine the derivative of the transformation $\zeta(z)$ that conformally maps a uniform flow in the upper half $z$-plane to the flow around a triangle with vertices at $\zeta = 0$, $\zeta = i$ and $\zeta = 1$. [4 marks]

**10**. Fluid flows with complex potential $w(z)$ in the semi-infinite strip $\mathrm{Re}\{z\} \geq 0$, $0 \leq \mathrm{Im}\{z\} \leq \pi$. The conformal map

$$\zeta(z) = \mathrm{e}^z$$

transforms this flow into a second one with potential $W(\zeta)$. Find the image of the strip under this conformal transformation, and hence the new flow domain. [12 marks]

**11**. The turning moment acting on an object with boundary $C$ due to a flow of a fluid with complex potential $w(z)$ is

$$M = -\frac{1}{2}\rho \operatorname{Re}\left\{ \oint_C z \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 \mathrm{d}z \right\},$$

where $\rho$ is the fluid density. The complex potential for flow with speed $U$ at an angle $\alpha$ to the positive real axis around an ellipse with circulation $\Gamma$, in a complex $\zeta$ plane, is

$$W(z(\zeta)) = U\left(z(\zeta)\mathrm{e}^{-i\alpha} + \frac{a^2}{z(\zeta)}\mathrm{e}^{i\alpha}\right) - \frac{i\Gamma}{2\pi}\mathrm{Log}(z(\zeta)),$$

where $z(\zeta)$ is the inverse of the conformal map $\zeta = z + \dfrac{c^2}{z}$ and $0 \leq c \leq a$.

(a) Give the lengths of the semi-major and semi-minor axes of the ellipse. [2 marks]

(b) Adapt the expression for $M$ above to the $\zeta$ plane, and hence calculate the turning moment on the ellipse. [10 marks]

End of exam paper