PRIFYSGOL ABERYSTWYTH UNIVERSITY

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 1 / SEMESTER 1 EXAMINATIONS

IONAWR / JANUARY 2026

MA37810 – Stochastic Models in Finance

The questions on this paper are written in English.

Amser a ganiateir - 2 awr **Time allowed - 2 hours**

- Gellir rhoi cynnig ar bob cwestiwn.
- Rhoddir mwy o ystyriaeth i berfformiad yn rhan B wrth bennu marc dosbarth cyntaf.
- Cyfrifianellau Casio FX-83 neu FX-85 YN UNIG a ganiateir.
- Darperir tablau ystadegol.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

- All questions may be attempted.
- Performance in section B will be given greater consideration in assigning a first class mark.
- Casio FX-83 or FX-85 calculators ONLY may be used.
- Statistical tables will be provided.
- Students may submit answers to this paper in either Welsh or English.

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Section A

1. Let $S(t)$ represent the price of a stock at time t, for $t = 0, 1, 2, \ldots$. At each time point the price either increases by one unit with probability p or decreases by one unit with probability $1 - p$.

For $i = 1, 2, \ldots$, let

$$X_i = \begin{cases} 1 & \text{if } S(i) = S(i-1) + 1 \\ -1 & \text{if } S(i) = S(i-1) - 1. \end{cases}$$

(a) Write down an expression for $S(t)$ in terms of $S(0)$ and the X_i's.

(b) If $S(0) = 10$ and $p = 0.25$, find

(i) $P(S(8) = 12 | S(5) = 11)$.

(ii) $\mathbb{E}(S(8) | S(5) = 11)$.

(iii) $\mathrm{Cov}(S(5), S(8))$.

[2 + 3 + 6 + 5 marks]

2. Let $S(t)$ be the price of a stock at time t for $t = 0, 1, 2, \ldots$. At each time point, the stock increases by 2 with probability 0.1, stays the same with probability 0.7, or decreases by 2 with probability 0.2.

Use an appropriate approximation to find the probability that after 800 time periods, the stock has dropped at least 100 units from its starting value. [7 marks]

3. (a) Suppose you wish to have a lump sum of £10000 available in 10 years' time. You invest £A per month at an annual interest rate of 5% compounded monthly. Find the value of A (indexing the months starting from 0).

You may use the fact that for $\beta \in \mathbb{R}, \beta \neq 1$,

$$\sum_{i=0}^{n-1} \beta^i = \frac{1 - \beta^n}{1 - \beta}.$$

(b) Suppose there is an annual rate of interest of 7.5%, compounded continuously. You wish to invest an initial sum £A such that you can withdraw £500 per month for 12 months and leave a lump sum of £500 after the final withdrawal. Find A, the value of your initial investment.

[6+5 marks]

4. Let $\{X(t) : t > 0\}$ be a Brownian motion with drift $\mu = 0.2$, variance $\sigma^2 = 1.6$ and $X(0) = 100$.

Find

(a) $P(X(10) > 110)$.

(b) $P(X(5) < X(0))$ and hence $P(X(10) < X(5) < X(0))$, i.e. the probability that $P(X(10) < X(5)$ and $X(5) < X(0))$.

[4+6 marks]

5. Let $\{S(t) : t \geq 0\}$ be a geometric Brownian motion process with drift parameter $\mu = 0.08$ and volatility parameter $\sigma = 0.4$. Find:

(a) $P(S(2.5) < S(0))$;

(b) $P(S(2.5) > S(1))$;

(c) K such that $P(S(1) > KS(0)) = 0.1$.

[3 + 3 + 4 marks]

6. In a four-horse race, the following odds are offered.

Horse	odds
1	Evens (1:1)
2	3:1
3	$k : 1$

(a) Find k such that there is no opportunity for arbitrage.

(b) Now assume $k = 6$. Let $x = (x_1, x_2, x_3)$ represent the betting strategy of betting £x_i on horse i, for $i = 1, 2, 3$. If $x_1 = 6$, write down three inequalities expressing x_3 in terms of x_2 to guarantee a positive return. Draw a feasibility diagram (with x_2 on the x-axis and x_3 on the y-axis) to show which combinations of x_2 and x_3 satisfy the three inequalities.

[2+8 marks]

7. Let $S(t)$, $t = 0, 1, 2, \ldots$ be the price of a stock at time t. The stock price is modelled as a multistep binomial branch model, where, at each time step, the price of the stock either increases by 10% with probability 0.5, or decreases by 6% with probability 0.5. Assume there is a fixed interest rate of 6% per time period. The Cox-Ross-Rubenstein formula states that the no-arbitrage price of an option to sell stock at a strike price of K after n time periods is given by

$$C_{N-A} = (1+R)^{-n} \sum_{m=\nu}^{n} \binom{n}{m} p_*^m (1-p_*)^{n-m} ((1+u)^m (1+d)^{n-m} S(0) - K),$$

where $p_* = (R-d)/(u-d)$ is the risk-neutral probability,

$$\nu = \min\{m \in 0, 1, \ldots, n : S_{n,m} \geq K\},$$

where $S_{n,m}$ is the stock price at time n if there are m upturns in the first n time steps, u and d are respectively the rates of return for an upturn and a downturn, and R is the interest rate. For a stock that starts at $S(0) =$ £150, calculate the no-arbitrage cost of an option to buy a unit of stock for £155 at $t = 3$.

[6 marks]

Section B

8. A stock is currently priced at £100. In one time period it will either increase by 20% (with probability 0.6) or decrease by 15% (with probability 0.4). An option is available to purchase a unit of stock at £110 at the end of the time period. If the interest rate is 3% per time period, find, showing your reasoning:

(a) The expectation price of the option.

(b) The no-arbitrage price of the option.

[9 + 12 marks]

9. A Brownian motion process with drift μ and variance σ^2 can be approximated (in discrete time) with a random walk. We split the interval $[0, t]$ into n discrete time steps, each of length $\Delta = t/n$. In each time increment, the process either increases by the amount $\sigma\sqrt{\Delta}$ with probability p, or decreases by the amount $\sigma\sqrt{\Delta}$ with probability $1-p$, where

$$p = \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{\Delta}\right).$$

Assume $X(0) = 0$, $\mu = 0.2$, $\sigma = 0.05$ and $\Delta = 0.0001$.

(a) Calculate the estimate of $P(X(0.001) \geq 0.004)$.

(b) For what value of k is the estimate of $P(X(0.001) \leq k)$ equal to $0.48^{10} + 10 \times 0.48^9 \times 0.52^1$? Explain your reasoning.

[10+5 marks]

10. Let $\{S(t) : t \geq 0\}$ be a geometric Brownian motion with drift μ and volatility σ. If $S(t)$ represents the price of a stock at time t, then we model the ratio

$$\frac{S(t+y)}{S(t)}$$

as a lognormal distribution, i.e.

$$\log\frac{S(t+y)}{S(t)} \sim N(\mu t, \sigma^2 t).$$

Suppose $S(0)$ is a given constant and an option is available to buy a stock at time t for £K per unit. If the fixed interest rate is r, compounded continuously, then the Black-Scholes formula states that the no-arbitrage price of the option is given by

$$C = S(0)\Phi(\omega) - Ke^{-rt}\Phi(\omega - \sigma\sqrt{t}),$$

where

$$\omega = \frac{rt + \frac{\sigma^2 t}{2} - \log\frac{K}{S(0)}}{\sigma\sqrt{t}},$$

and

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,\mathrm{d}x$$

is the cumulative density of a standard $N(0,1)$ random variable.

(a) If the starting price of a stock is £25, find the no-arbitrage price of an option to buy stock for £26 after 18 months if $\mu = 0.01$, $\sigma = 0.04$ and the annual interest rate is 0.06, compounded continuously.

(b) Show that if the strike price, K, is given by

$$K = S(0)\exp((r + \sigma^2/2)t)$$

then the no-arbitrage price can be written

$$C = \frac{S(0)}{2} - Ke^{-rt} + Ke^{-rt}\Phi(\sigma\sqrt{t}).$$

[7+7 marks]