PRIFYSGOL ABERYSTWYTH UNIVERSITY

**ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS**

**ARHOLIADAU SEMESTER 1 / SEMESTER 1 EXAMINATIONS**

**IONAWR / JANUARY 2026**

# MAM3420 – Advanced Mathematical Methods in the Theory of Elasticity

The questions on this paper are written in English.

**Amser a ganiateir - 2 awr**

- Gellir rhoi cynnig ar bob cwestiwn.
- Rhoddir mwy o ystyriaeth i berfformiad yn Rhan B wrth bennu marc dosbarth cyntaf.
- Cyfrifianellau Casio FX-83 neu FX-85 YN UNIG a ganiateir.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

**Time allowed - 2 hours**

- All questions may be attempted.
- Performance in Section B will be given greater consideration in assigning a first class mark.
- Casio FX-83 or FX-85 calculators ONLY may be used.
- Students may submit answers to this paper in either Welsh or English.

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**Supplementary material**

**The strain - displacement and rotation - displacement relations** for a symmetric infinitesimal strain tensor $\varepsilon_{ij}$, rotation tensor $\Omega_{ij}$, and displacement $u_i$ on and within an elastic body take the form

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \quad \Omega_{ij} = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right).$$

**Hooke's law** relating the stress field $\sigma_{ij}$ and the strain field $\varepsilon_{ij}$ within an elastic, homogeneous and isotropic material takes the form

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}, \quad \varepsilon_{ij} = \frac{1}{2\mu}\left(\sigma_{ij} - \frac{\lambda}{3\lambda+2\mu}\sigma_{kk}\delta_{ij}\right),$$

where $\lambda$, $\mu$ are Lamé coefficients.

**Cauchy's equations** for a body at equilibrium state that the stress field $\sigma_{ij}$ and the body forces $F_i$ experienced by an elastic body must be related as

$$\frac{\partial\sigma_{ij}}{\partial x_j} + F_i = 0.$$

**Differential operators:** In spherical coordinates $(r, \theta, \phi)$ acting on the orthonormal basis $(\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi)$, we have:

$$\text{grad}(f) = \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{e}_\theta + \frac{1}{r\sin(\theta)}\frac{\partial f}{\partial\phi}\hat{e}_\phi,$$

$$\text{div}(\mathbf{u}) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right) + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)u_\theta\right) + \frac{1}{r\sin(\theta)}\frac{\partial u_\phi}{\partial\phi},$$

$$\text{curl}(\mathbf{u}) = \frac{1}{r\sin(\theta)}\left[\frac{\partial}{\partial\theta}\left(\sin(\theta)u_\phi\right) - \frac{\partial u_\theta}{\partial\phi}\right]\hat{e}_r + \frac{1}{r}\left[\frac{1}{\sin(\theta)}\frac{\partial u_r}{\partial\phi} - \frac{\partial}{\partial r}\left(ru_\phi\right)\right]\hat{e}_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(ru_\theta\right) - \frac{\partial u_r}{\partial\theta}\right]\hat{e}_\phi.$$

The **Airy stress function** $\phi$ in polar coordinates $(r, \theta)$ is a scalar function such that

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}, \quad \sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2}, \quad \sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right).$$

**The Neumann iterative scheme** for a Volterra-type integral equation of the second kind with kernel $K(x, y)$ is given by

$$\mathcal{K}_0(x, y) = K(x, y), \quad \mathcal{K}_n(x, y) = \int_y^x K(x, s)\mathcal{K}_{n-1}(s, y)\,\mathrm{d}s,$$

with the resolvent kernel

$$R_n(x, y; \lambda) = \sum_{i=0}^{n}\lambda^i\mathcal{K}_i(x, y), \quad R(x, y; \lambda) = \lim_{n\to\infty} R_n(x, y; \lambda).$$

**The Laplace transform** $\mathcal{L}$ applied to the second derivative of a function $u(t)$ has the property

$$\mathcal{L}\left\{u''\right\}(s) = s^2\mathcal{L}\left\{u\right\}(s) - su(0) - u'(0), \quad \mathbb{R}[s] > 0.$$

**Notable transforms:** You may use without proof that

$$\mathcal{L}\left\{\delta(t-a)\right\}(s) = e^{-as}, \quad a > 0$$

$$\mathcal{L}\left\{\cos(t)\right\}(s) = \frac{s}{s^2+1}, \quad \mathbb{R}[s] > 0$$

$$\mathcal{L}\left\{\sin\left[a(t-b)\right]H(t-b)\right\}(s) = \frac{ae^{-bs}}{s^2+a^2}, \quad \mathbb{R}[s] > 0$$

$$\mathcal{L}\left\{t\sin(t)\right\}(s) = \frac{2s}{(s^2+1)^2} \quad \mathbb{R}[s] > 0$$

## Section A

**1**. Consider a homogeneous, isotropic, elastic cube defined within the region $x, y, z \in [0, 1]$. The displacement field within and on the cube is given by

$$\mathbf{u} = \begin{bmatrix} s(2x^2 + 3ys) \\ s(y^2 + z - 3xs) \\ s(y - z^2) \end{bmatrix},$$

where $0 < s \ll 1$ is a small constant. The equations needed to answer this question are provided in the supplementary material.

(a) Determine the infinitesimal strain field $\boldsymbol{\varepsilon}$ and rotation tensor $\boldsymbol{\Omega}$ within the elastic body. [8 marks]

(b) Without evaluating them, state whether this infinitesimal strain field $\boldsymbol{\varepsilon}$ will satisfy the Saint-Venant compatibility equations. Justify your answer. [3 marks]

(c) State whether an infinitesimal strain approximation is suitable in this case. Explain your reasoning. [2 marks]

**2**. Use the separable kernel method to solve

$$u(x) = 12x^4 + \lambda \int_0^1 x^8 y u(y)\, \mathrm{d}y,$$

stating any conditions on the constant $\lambda \in \mathbb{R}$. [6 marks]

**3**. A unit cube of jelly sits on a flat plate. It is subject to uniform atmospheric pressure $p$ on its faces. The cube is pulled down by gravity with the medium experiencing force per unit volume $F_z = -\rho g$, where $g$ is the gravitational constant and $\rho$ is the density of the medium. No other forces are acting on or within the jelly. The system is assumed to be at equilibrium. The jelly is approximated as a homogeneous, isotropic, elastic body.

All equations needed to answer this question are given in the supplementary material. Part (b) can be answered without completing part (a).

(a) (i) State the body forces $\boldsymbol{F}$ acting within the cube. [2 marks]

(ii) Assume that the cube is in its principal coordinate system. Use Cauchy's equations to demonstrate that the stress tensor in the cube is in the form

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx}(y, z) & 0 & 0 \\ 0 & \sigma_{yy}(x, z) & 0 \\ 0 & 0 & \rho g z + f(x, y) \end{bmatrix},$$

for some unknown function $f$. [6 marks]

(b) (i) State the normal vectors and tractions on the following faces of the cube

- $\{x = 1;\ y, z \in [0, 1]\}$;
- $\{y = 1;\ x, z \in [0, 1]\}$;
- $\{z = 1;\ x, y \in [0, 1]\}$.

[3 marks]

(ii) Hence, determine the unique stress tensor for the jelly cube. [6 marks]

**4**. The Neumann iterative scheme for a Volterra-type integral equation of the second kind is provided in the supplementary material.

(a) Verify that the Volterra-type integral equation

$$u(x) = f(x) + \lambda \int_0^x x^4 y^2 u(y)\,\mathrm{d}y, \quad \lambda \in \mathbb{R},$$

has its $n^{\text{th}}$ term of Neumann iteration given by $\mathcal{K}_n(x,y) = \dfrac{x^4y^2}{n!}\left(\dfrac{x^7-y^7}{7}\right)^n$. [5 marks]

(b) Hence, determine the resolvent kernel for this integral equation. State the solution $u(x)$ in terms of the resolvent kernel. [3 marks]

**5**. The Cauchy stress tensor at a point in a homogeneous, isotropic, elastic medium is given by

$$\boldsymbol{\sigma} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}.$$

(a) Show that the principal values are $\lambda_1 = 4$, $\lambda_2 = 1$, $\lambda_3 = -2$. Determine the associated principal axes, $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, $\boldsymbol{n}_3$, for the stress $\boldsymbol{\sigma}$ at this point. [10 marks]

(b) State the stress tensor at this point when transformed to the principal coordinate system. [1 mark]

(c) Show that the infinitesimal strain tensor $\boldsymbol{\varepsilon}$ experienced at this point in the elastic medium, in its principal coordinate system, is given by

$$\boldsymbol{\varepsilon} = \frac{1}{2\mu}\begin{bmatrix} 4 - \dfrac{3\lambda}{3\lambda+2\mu} & 0 & 0 \\ 0 & 1 - \dfrac{3\lambda}{3\lambda+2\mu} & 0 \\ 0 & 0 & -2 - \dfrac{3\lambda}{3\lambda+2\mu} \end{bmatrix},$$

where $\lambda, \mu > 0$ are the Lamé coefficients for the medium. [4 marks]

**6**. Consider the boundary value problem

$$\begin{cases} u'' + 9u = f(x), \quad 0 < x < \dfrac{\pi}{2}, \\ u(0) = u\left(\dfrac{\pi}{2}\right) = 0, \end{cases}$$

where $f : \left[0, \dfrac{\pi}{2}\right] \to \mathbb{R}$ is a continuous function.

(a) State the definition of the Green's function corresponding to this boundary value problem, including the boundary conditions it must satisfy. [3 marks]

(b) Determine the Green's function associated with the boundary value problem. [8 marks]

## Section B

**7**. The bathysphere of a submersible designed to reach the bottom of the Mariana Trench can be approximated as a hollow, homogeneous and isotropic elastic sphere. The sphere has inner radius $r_0$ and outer radius $r_1$. It is subject to uniform internal pressure $p_0$, and uniform external pressure $p_1$. We assume that the sphere is at equilibrium, and consider the system in spherical coordinates $(r, \theta, \phi)$. The Navier-Lamé equations for the bathysphere are given by

$$2(1-\nu)\mathsf{grad}\,(\mathsf{div}\,(\boldsymbol{u})) - (1-2\nu)\mathsf{curl}\,(\mathsf{curl}\,(\boldsymbol{u})) = 0,$$

where $\nu$ is the Poisson's ratio. The form of the differential operators is provided in the supplementary material. Part (c) can be completed independently of (a)-(b).

(a) Explain how the displacement $\boldsymbol{u}$, and its radial component $u_r$, depend on the angles $(\theta, \phi)$. Hence, simplify the Navier-Lamé equations for this case. [6 marks]

(b) Solve the resulting equation to demonstrate that the radial displacement can be represented in the form

$$u_r = Ar + \frac{B}{r^2},$$

for some constants $A, B$. [2 marks]

(c) The yield stress criterion provides a condition which, when satisfied, indicates that the medium will exceed the yield point of its constituent material. For the hollow sphere with material dependent yield stress $Y$ it can be formulated as

$$|\sigma_{\theta\theta} - \sigma_{rr}| = Y.$$

For the hollow sphere, it can be shown that the stress field has components

$$\sigma_{rr} = \frac{1}{r_1^3 - r_0^3}\left[\frac{r_0^3 r_1^3(p_0+p_1)}{r^3} - (p_1 r_1^3 + p_0 r_0^3)\right],$$

$$\sigma_{\theta\theta} = \sigma_{\phi\phi} = -\frac{1}{r_1^3 - r_0^3}\left[\frac{r_0^3 r_1^3(p_0+p_1)}{2r^3} + (p_1 r_1^3 + p_0 r_0^3)\right].$$

Assume that the bathysphere has inner radius $r_0 = 1$ m, and internal pressure identical to that at the surface $p_0 = 10^5$ Pa. The sphere is comprised of titanium alloy, which has a yield strength of $Y = 8.3 \times 10^8$ Pa. Determine the minimum outer radius $r_1$ needed to prevent the sphere exceeding the yield point when diving to the bottom of the Mariana Trench, where the external pressure is $10^8$ Pa. Provide your answer in the form of a cube root. [10 marks]

**8**. A homogeneous, isotropic, elastic half-space is subject to vertical loading of a magnitude $P$ at a single point on the surface. The resulting stress field can be generated in polar coordinates $(r, \theta)$ from the Airy stress function

$$\phi(r, \theta) = -\frac{P}{\pi} r\theta \sin(\theta).$$

The equations needed to solve this question are provided in the supplementary material. Note that all of parts (a)-(c) can be completed independently of one another.

(a) Demonstrate that the stress field induced by the point loading has components

$$\sigma_{rr} = -\frac{2P\cos(\theta)}{\pi r}, \quad \sigma_{\theta\theta} = \sigma_{r\theta} = 0.$$

[4 marks]

(b) For a plane strain system in polar coordinates, the radial and hoop stress, infinitesimal strain and displacement are related as

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} = \frac{1+\nu}{E}\left[(1-\nu)\sigma_{rr} - \nu\sigma_{\theta\theta}\right],$$

$$\varepsilon_{\theta\theta} = \frac{1}{r}\left[\frac{\partial u_\theta}{\partial \theta} + u_r\right] = \frac{1+\nu}{E}\left[(1-\nu)\sigma_{\theta\theta} - \nu\sigma_{rr}\right],$$

where $E$ is the Young's modulus, and $\nu$ the Poisson's ratio. Demonstrate that the displacement field generated by the point loading takes the form

$$u_r = -\frac{2P(1-\nu^2)}{\pi E}\cos(\theta)\log(r) + f(\theta),$$

$$u_\theta = \frac{2P(1+\nu)}{\pi E}\sin(\theta)\left[\nu + (1-\nu)\log(r)\right] - \int f(\theta)\,\mathrm{d}\theta + g(r),$$

for some unknown functions $f(\theta)$ and $g(r)$. [5 marks]

(c) By considering the shearing strains $\varepsilon_{r\theta}$ in the absence of arbitrary rigid displacements, we find that $g(r) = 0$ while $f(\theta)$ is the solution of

$$\frac{\mathrm{d}f(\theta)}{\mathrm{d}\theta} + \frac{2P(1+\nu)(1-2\nu)}{\pi E}\sin(\theta) + \int f(\theta)\,\mathrm{d}\theta = 0.$$

By converting this integro-differential equation to a second order differential equation, determine the form of $f(\theta)$. [6 marks]

**9**. Consider the nonlinear Fredholm-type integral equation

$$u(x) = x - \lambda \int_0^1 x u^2(y)\,\mathrm{d}y, \quad \lambda \in \mathbb{R}, \quad 0 < x < 1.$$

(a) Briefly outline two methods for obtaining the solutions of this integral equation. [5 marks]

(b) Using either of your proposed methods, determine the solutions of the nonlinear integral equation. State any conditions on the constant $\lambda$. [12 marks]