PRIFYSGOL ABERYSTWYTH UNIVERSITY

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 2 / SEMESTER 2 EXAMINATIONS

MAI / MAY 2026

MA11110 - Mathematical Analysis

The questions on this paper are written in English.

Amser a ganiateir - 2 awr

- Rhoddir marciau llawn am atebion cyflawn i bob cwestiwn yn rhan A ac i dri cwestiwn yn rhan B.
- Yn rhan B, rhoddir credyd am y tri ateb gorau.
- Ni cheir defnyddio cyfrifianellau.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

Time allowed - 2 hours

- Full marks will be given for complete answers to all questions in section A and to three questions in section B.
- In section B, credit will be given for the best three answers.
- Calculators are not permitted.
- Students may submit answers to this paper in either Welsh or English.

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Section A

1. Let $\epsilon > 0$ be given. Find $\delta > 0$ such that for $x \in \mathbb{R}$ satisfying $|x| < \delta$, we have $|x^5 + 4x| < \epsilon$. [5 marks]

2. Suppose that x satisfies $|x| < 1$. Prove that

$$\text{(a) } |(x+1)^2(x+5)| < 24; \text{ (b) } \left|\frac{(x+2)(x+3)}{(x-2)(x-4)}\right| < 4.$$

[5 marks]

3. Prove that the set $[0, 7)$ has a minimum element, but no maximum. [5 marks]

4. Prove that the set $\{x^2 + 4x + 7 : x \in \mathbb{R}\}$ is bounded below, but unbounded above. [5 marks]

5. Working directly with the (N, ϵ)-definition of convergence, prove that

$$\lim_{n\to\infty} \frac{2n+3}{10n+1} = \frac{1}{5}.$$

[5 marks]

6. For $n \in \mathbb{N}$, define $a_n = \cos(n\pi/3)$. Prove that the sequence $(a_n)_{n=1}^{\infty}$ does not converge. [5 marks]

7. Classify the following series as either convergent or non-convergent:

$$\text{(a) } \sum_{n=0}^{\infty} \frac{(-3)^{n+2}}{5^n}; \text{ (b) } \sum_{n=0}^{\infty} \frac{2^n}{47}.$$

If the series is convergent, find its sum. [5 marks]

8. Use L'Hôpital's rule to evaluate the following limits:

$$\text{(a) } \lim_{x\to 2} \frac{x^3 - 7x^2 + 10x}{x^2 + x - 6}; \text{ (b) } \lim_{x\to 0} \frac{\exp(x) - x - 1}{2x^2}.$$

[5 marks]

SECTION B BEGINS ON THE NEXT PAGE

Section B

9. (a) Classify the following subsets of $\mathbb{R}$ as bounded or unbounded. If the set is bounded, write down the supremum and infimum.

(i) $(6, 7]$; (ii) $\{x : |x - 2| < 7\}$;
(iii) $\{x : x^2 - 4x + 8 \geq 1\}$; (iv) $\{1 - 1/n : n \in \mathbb{N} \text{ and } n \text{ is prime}\}$.

(There is no need to provide proofs of your assertions.) [7 marks]

(b) Let E be a non-empty subset of $\mathbb{R}$ that is bounded above. Prove that for every $\epsilon > 0$, there exists $x \in E$ such that

$$\sup E - \epsilon < x \leq \sup E,$$

where $\sup E$ denotes the supremum of E. [4 marks]

(c) Let $A \subset \mathbb{R}$ be non-empty and bounded. Define $B, C \subset \mathbb{R}$ by:

$$B = \{4x - 3 : x \in A\};\ C = \{-2x + 6 : x \in A\}.$$

Prove that

$$\sup B = 4 \sup A - 3;\ \sup C = -2 \inf A + 6,$$

where $\inf D$ denotes the infimum of D. [4,5 marks]

10. (a) Working directly with the (N, ϵ)-definition of convergence, prove that:

$$\text{(i)} \lim_{n\to\infty} \frac{n^2 \cos(6n)}{2n^3 - 1} = 0;\ \text{(ii)} \lim_{n\to\infty} \sqrt{n + 5} - \sqrt{n + 2} = 0.$$

[4,5 marks]

(b) Let $(a_n)_{n=1}^{\infty}$ be a sequence defined recursively by $a_1 = a_2 = 1$, and for $n \in \mathbb{N}$, $n \geq 2$, $a_{n+1} = a_n + 2a_{n-1}$. Given that $\lim_{n\to\infty} a_{n+1}/a_n = a$ for some $a \in \mathbb{R}$, determine a. [6 marks]

(You may use any algebra of limits results you know without proof.)

(c) Evaluate

$$\lim_{n\to\infty} \frac{(n + 2)!(2n + 1)!}{(3n + 2)!}.$$

[5 marks]

(You may quote any relevant results you know without proof.)

11. (a) Classify the following sequences as either monotone increasing, monotone decreasing, or not monotonic:

$$\text{(i) } ((-1)^n n^3)_{n=1}^{\infty};\ \text{(ii) } \left(\exp\left(\frac{2n+1}{5n-3}\right)\right)_{n=1}^{\infty};\ \text{(iii) } \left(\frac{3^n}{2n}\right)_{n=1}^{\infty}.$$

(There is no need to provide proofs of your assertions.) [6 marks]

(b) A sequence $(a_n)_{n=1}^{\infty}$ of real numbers is defined recursively by

$$a_1 = 1,\ a_{n+1} = \frac{1}{5}(a_n + 2) \text{ for } n \in \mathbb{N}.$$

(i) Calculate a_2 and a_3. [2 marks]

(ii) Prove by induction that $a_n > 1/2$ for each $n \in \mathbb{N}$. [4 marks]

(iii) Show that $(a_n)_{n=1}^{\infty}$ is a monotone decreasing sequence. [4 marks]

(iv) Explain why $\lim_{n\to\infty} a_n$ exists, and determine its value. [4 marks]

12. Determine the convergence or otherwise of the following series, stating clearly which convergence test you are using and your reasoning:

$$\text{(a) } \sum_{n=1}^{\infty} \frac{(n+1)^3}{\exp(6n)};\ \text{(b) } \sum_{n=1}^{\infty} \sqrt{n^2+7} - n;\ \text{(c) } \sum_{n=1}^{\infty} \frac{(-1)^n}{n+6};\ \text{(d) } \sum_{n=1}^{\infty} \frac{(n+2)(n+3)}{n^2+6}.$$

[5,6,5,4 marks]

13. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 + 2 & \text{if } x \leq -1, \\ x + 5 & \text{if } x > -1. \end{cases}$$

(i) Sketch the graph of f. [2 marks]

(ii) Demonstrate that

$$\lim_{x\to\infty} f(x) = +\infty.$$

[3 marks]

(iii) Prove that f is discontinuous at $x = -1$. [4 marks]

(iv) Suppose $x < -1$. Prove that f is continuous at x. [5 marks]

(b) Show that the equation $2x^{11} + 4x^7 + 13x - 4 = 0$ has exactly one real root, stating clearly any theorems that you use. [6 marks]

END OF EXAM PAPER