PRIFYSGOL ABERYSTWYTH UNIVERSITY

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 2 / SEMESTER 2 EXAMINATIONS

MAI / MAY 2026

MA11210 – Differential Equations

The questions on this paper are written in English.

Amser a ganiateir - 2 awr

- Rhoddir marciau llawn am atebion cyflawn i bob cwestiwn yn Rhan A ac i dri cwestiwn yn Rhan B.
- Yn Rhan B, rhoddir credyd am y tri ateb gorau.
- Ni cheir defnyddio cyfrifianellau.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

Time allowed - 2 hours

- Full marks will be given for complete answers to all questions in Section A and to three questions in Section B.
- In Section B, credit will be given for the best three answers.
- Calculators are not permitted.
- Students may submit answers to this paper in either Welsh or English.

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Section A

1. Write down the order and degree of the following differential equations and state whether or not they are linear:

(a) $\left(\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}\right)^2 + 2\dfrac{\mathrm{d}y}{\mathrm{d}x} + y^2 = x;$

(b) $\left(\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)^3 + y = 2;$

(c) $\dfrac{\mathrm{d}^3x}{\mathrm{d}t^3} + t\dfrac{\mathrm{d}x}{\mathrm{d}t} - 1 = 0;$

(d) $\dfrac{\mathrm{d}y}{\mathrm{d}x} + e^x y = x;$

(e) $\sin x\dfrac{\mathrm{d}y}{\mathrm{d}x} = 7(y - x - 1).$

[5 marks]

2. Find the solution of the following differential equation that satisfies $y(0) = 0$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3e^{-x} + 6x + 2\cos(2x).$$

[5 marks]

3. Solve the following linear first order differential equation given that $y(1) = 3$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2}{x}y = 10x.$$

[5 marks]

4. Find an explicit particular solution of the following differential equation satisfying the condition $y(0) = 1$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{e^x + 2}{6y^2}.$$

[5 marks]

5. Show that the following differential equation is homogeneous and find its general solution:

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = x + y.$$

[5 marks]

6. Find the general solution of the following second order differential equation:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 12e^x.$$

[5 marks]

7. Find the solution $y = y(t)$ of the following differential equation that satisifies the specified initial conditions:

$$\ddot{y} + 4y = 8e^{2t}; \quad y(0) = 1, \quad \dot{y}(0) = 4.$$

[5 marks]

8. Find the explicit general solution of the following Bernoulli equation:

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y + 10x^2y^2 = 0.$$

[5 marks]

Section B

9. Find the general solutions of the following differential equations:

(a) $y'' - 3y' - 4y = 12e^{3x} + 4x^2 + 6x + 2$; [10 marks]

(b) $y'' - 6y' + 9y = 4e^{3x} + 50\cos(x)$. [10 marks]

10. (a) By considering the derivative of $u = e^y \sin(2x)$ or otherwise, find the explicit general solution of the following differential equation:

$$e^y\left(\cos(2x) + \sin(2x)\right) + \frac{e^y}{2}\sin(2x)\frac{\mathrm{d}y}{\mathrm{d}x} - 2x = 0.$$

[10 marks]

(b) Find the (implicit) general solution of the following non-linear differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x + y + 3}{x - y + 3}.$$

[10 marks]

11. Leaving the solution in as simple a form as possible, solve

$$y'' + y = \sec x,$$

for $x \in (-\pi/2, \pi/2)$, subject to the boundary conditions $y(0) = 1$ and $y'(0) = 1$. [20 marks]

Section B continues on next page

12. A cup of coffee is placed outside. The ambient temperature is constant at $20°\mathrm{C}$. Newton's law of cooling gives

$$\frac{\mathrm{d}T_s}{\mathrm{d}t} = -k_0(T_s - 20), \qquad (*)$$

where $T_s(t)$ is the temperature (°C) of the coffee at time t (in minutes) in still air, and $k_0 > 0$ is a constant heat-transfer coefficient.

Suppose now that wind speed varies periodically in time, altering the rate of heat loss. The effective heat-transfer coefficient is

$$k(t) = k_0\left(1 + \frac{1}{5}\sin t\right),$$

and so, letting $T_w(t)$ denote the temperature of the coffee on a windy day,

$$\frac{\mathrm{d}T_w}{\mathrm{d}t} = -k_0\left(1 + \frac{1}{5}\sin t\right)(T_w - 20). \qquad (\dagger)$$

(a) Find the explicit particular solution of $(\dagger)$ subject to the initial condition $T_w(0) = 90$. [8 marks]

(b) Clearly explaining your reasoning, determine $\lim_{t\to\infty} T_w(t)$ and give a physical interpretation of the result. [5 marks]

(c) Without solving any further differential equations, state the solution of $(*)$ with the initial condition $T_s(0) = 90$. Suppose one cup of coffee is placed in still air and another in the breeze at the same time. What can be said about which coffee is hotter, and when? Justify your answer mathematically. [7 marks]

13. A large open tank has constant horizontal cross-sectional area A m^2. At the bottom of the tank there is a small circular hole of area a m^2, where a is much less than A.

Let $h(t)$ denote the depth of water (in metres) in the tank at time t seconds.

According to Torricelli's law, the speed of water leaving through the hole is

$$v = \sqrt{2gh},$$

where g is the acceleration due to gravity in $\mathrm{m/s^2}$. Accordingly, the volumetric outflow rate is

$$Q_{\text{out}} = a\sqrt{2gh}.$$

Initially, the tank contains water of depth $h_0 > 0$, and no water flows into the tank.

(a) Using conservation of volume, derive a separable first-order differential equation satisfied by $h(t)$ and state an appropriate initial condition. [5 marks]

(b) Solve the initial value problem obtained in part (a) explicitly to obtain $h(t)$. [7 marks]

(c) Find the time T at which the tank empties (that is, when $h(T) = 0$). Express your answer in terms of A, a, g, and h_0. [3 marks]

(d) What happens to $\dfrac{\mathrm{d}h}{\mathrm{d}t}$ as $t \to T$? Justify your answer mathematically. [5 marks]

END OF PAPER