PRIFYSGOL ABERYSTWYTH UNIVERSITY

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 2 / SEMESTER 2 EXAMINATIONS

MAI – MEHEFIN / MAY – JUNE 2026

MA11310 – Statistics

The questions on this paper are written in English.

Amser a ganiateir - 2 awr

- Rhoddir marciau llawn am atebion cyflawn i bob cwestiwn yn Rhan A ac i dri cwestiwn yn Rhan B.
- Yn Rhan B, rhoddir credyd am y tri ateb gorau.
- Cyfrifianellau Casio FX-83 neu FX-85 YN UNIG a ganiateir.
- Darperir tablau ystadegol.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

Time allowed - 2 hours

- Full marks will be given for complete answers to all questions in Section A and to three questions in Section B.
- In Section B, credit will be given for the best three answers.
- Casio FX-83 or FX-85 calculators ONLY may be used.
- Statistical tables will be provided.
- Students may submit answers to this paper in either Welsh or English.

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Section A

1. The random variable X has mean $1/7$ and standard deviation $1/4$.

(a) Find the values of (i) $\mathbb{E}[28X - 6]$ (ii) $\mathrm{Var}(28X - 6)$.

(b) Between what limits does the bulk of values of $28X - 6$ lie? [5 marks]

2. The random variable X has a Binomial, $\mathrm{Bin}(n, p)$, distribution with mean 308 and variance 187. Find (a) the values of n and p, (b) $\mathbb{E}[X(X-1)]$. [5 marks]

3. In the following cases, state whether the Binomial model $\mathrm{Bin}(n, p)$ might be valid for the random variable X. If yes, give n and p. If not, suggest a sensible alternative.

(a) 80% of appliances from a certain manufacturer are functional, the rest being faulty. The appliances in a large batch are tested one by one until 10 functional appliances are found; X is the number of appliances tested.

(b) 20% of batteries from a certain supplier are known to last longer than 1200 hours. Among 30 randomly selected batteries, X is the number of those that last less than 1200 hours.

(c) Light bulbs of a certain type are known to last 3000 hours on average, with standard deviation 50 hours. X is the sum of the lifetimes of 100 randomly selected lightbulbs of this type. [5 marks]

4. Use Statistical Tables to find (a) $\mathbb{P}(2 \leq X < 7)$ if $X \sim \mathrm{Bin}(100, 0.05)$, and (b) $\mathbb{P}(Y > 92)$ if $Y \sim \mathrm{Bin}(100, 0.96)$. [5 marks]

5. A statistic X has probability density function (pdf)

$$f(x) = 4\theta^{-4}x^3, \quad 0 < x < \theta,$$

where $\theta > 0$ is an unknown parameter. Show that $\mathbb{P}(0.1\theta < X < 0.9\theta) = 0.656$, and hence deduce a 65.6% confidence interval for θ. [5 marks]

6. A company owns a chain of five coffee shops, the daily revenues of which are independent and Normally distributed with means 320, 450, 112, 640, 270 (£), and standard deviations 20, 50, 40, 50, 30 (£), respectively.

(a) Find the mean and standard deviation of the total daily revenue of the chain.

(b) Find the probability that the total revenue on one day is less than 1600£. [5 marks]

7. Let $\mathrm{Var}(U) = 2$, $\mathrm{Var}(V) = 5$ and $\mathrm{Var}(U - 3V) = 29$. Find (a) covariance between U and V, and (b) $\mathrm{Cov}(U, 2U + V)$. [5 marks]

8. When a beam of laser light is pointed at a photon detector, the photons in the beam arrive at the detector randomly at an average rate of λ per microsecond.

(a) State the distribution of the number of photons detected in 5 microseconds.

(b) Carry out a test of hypotheses $H_0 : \lambda = 3$ versus $H_1 : \lambda > 3$, given that the detector was switched on for 5 microseconds and 22 photons were detected. State the P-value and your conclusion using 5% significance level. [5 marks]

Section B

9. (a) Evaluate $\mathbb{P}(X = k+1)/\mathbb{P}(X = k)$ and hence find the mode(s) of X when

(i) $X \sim \mathrm{Pasc}(10, \frac{1}{2})$; [6 marks]

(ii) $\mathbb{P}(X = k) = \dfrac{k}{20}\left(\dfrac{4}{5}\right)^k$, $k = 1, 2, \ldots$. [4 marks]

(b) 20% of a large electorate support candidate A in an election, and 50% support candidate B. Let X and Y be the numbers of supporters of A and B, respectively, in a poll involving 800 randomly chosen members of the electorate.

(i) Identify the distributions of X, Y, and $X + Y$; [3 marks]

(ii) Find the variances of X, Y, and $X + Y$; [3 marks]

(iii) Find the correlation between X and Y. [4 marks]

10. The proportion p of appliances produced by a manufacturer are functional.

(a) Suppose $p = 0.93$. Calculate the probability that out of 20 appliances tested, more than 3 fail to function, using the following methods:

(i) the exact Binomial distribution; [4 marks]

(ii) the Normal approximation; [5 marks]

(iii) the Poisson approximation. [3 marks]

Make sure to name the relevant variable, and in each case (i)-(iii), state clearly the relevant distribution, and find the probability using Statistical Tables.

(b) The manufacturer claims that 93% of the appliances are functional, but you suspect that less of them are. Twenty appliances were tested, and 16 were found functional.

(i) Based on this information, carry out a hypothesis test to assess the claim at 5% significance level, using the result in part (a.i). To get full marks, carefully formulate the relevant steps. [6 marks]

(ii) Do either of the two approximations in part (a) give the correct conclusion? Justify your reasoning based on your results in part (a). [2 marks]

11. When a beam of laser light is pointed at a photon detector, the photons in the beam arrive at the detector screen randomly at an average rate of λ per microsecond. Each arrival results in a detection event.

(a) Let $\lambda = 3$, and consider the detection of photons in two non-overlapping observation periods of 10 and 20 microseconds, respectively. Let N_1 and N_2 be the resulting numbers of detections.

(i) State the distributions of N_1 and N_2. [3 marks]

(ii) Find the probability that more than 20 but less than 30 photons are detected in the first observation period. [3 marks]

(iii) Calculate the correlation between $N_1 + N_2$ and $N_1 - N_2$. [4 marks]

(b) Let T be the time at which the first photon is detected (measured from the time at which the detector is switched on). State the distribution of T and explain why T is an unbiased estimator of the parameter $\phi = 1/\lambda$. [2 marks]

(c) According to the manufacturer, the laser produces photons with rate $\lambda = 3$. Given that the first photon was detected 0.04 microseconds after switching on the detector, is it reasonable to suspect that the rate is higher than claimed? Carry out an appropriate test: state the hypotheses in terms of the parameter ϕ in (b), write down the pdf of the test statistic assuming the null hypothesis, calculate the P-value, and state your conclusion using 5% significance level. [8 marks]

12. Random variable X has pdf

$$f(x) = \begin{cases} 2\theta^{-1}x, & 0 < x < \theta, \\ 2(1-\theta)^{-1}(1-x), & \theta < x < 1, \end{cases}$$

where $0 < \theta < 1$ is an unknown parameter.

(a) Show that $f(x)$ is a valid pdf, explaining, in particular, where you use the condition $0 < \theta < 1$. [5 marks]

(b) Show that $\mathbb{E}[X] = (1+\theta)/3$. [3 marks]

(c) Find an unbiased estimator T of θ based on X. [3 marks]

(d) A random sample $X_1, \ldots, X_{50}$ is drawn from the distribution of X.

(i) State an unbiased estimator $\overline{T}$ of θ based on the sample mean $\overline{X}$. [1 mark]

(ii) When $\theta = \frac{1}{2}$, use the Central Limit Theorem to identify the approximate distribution of $\overline{T}$. [8 marks]

13. (a) Let X, Y, and V be jointly Normal random variables with respective means 2, -2, 0, and standard deviations 5, 5, 7. The correlation between X and Y is $-1/2$, the correlation between X and V is $1/2$, and the correlation between Y and V is $-2/3$.

(i) Calculate $\mathrm{Var}(X + 2Y)$. [3 marks]

(ii) Identify the distribution of $X + 2Y$. [2 marks]

(iii) Calculate $\mathrm{Cov}(X + 2Y, 3V)$. [3 marks]

(iv) Identify the distribution of $X + 2Y + 3V$, and hence calculate the probability that $X + 2Y + 3V > 0$. [5 marks]

(b) The length of a certain type of fish in a particular lake is known to be Normally distributed with unknown mean μ and known standard deviation 5 cm. Consider the test of hypothesis $H_0 : \mu = 20$ (cm) against the alternative $H_1 : \mu > 20$, based on a catch of 10 fish of average length 22.7 cm.

(i) Explain (in a few words) what the *population* is in this context. [1 mark]

(ii) Find the distribution of the relevant test statistic, assuming that the null hypothesis is true. [2 marks]

(iii) Find the P-value, and state the conclusion of the hypothesis test using 5% significance level. [4 marks]