PRIFYSGOL ABERYSTWYTH UNIVERSITY

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 2/ SEMESTER 2 EXAMINATIONS

MAI / MAY 2026

MA21410 - Linear Algebra

The questions on this paper are written in English.

Amser a ganiateir - 2 awr

- Gellir rhoi cynnig ar bob cwestiwn.
- Rhoddir mwy o ystyriaeth i berfformiad yn rhan B wrth bennu marc dosbarth cyntaf.
- Cyfrifianellau Casio FX-83 neu FX-85 YN UNIG a ganiateir.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

Time allowed - 2 hours

- All questions may be attempted.
- Performance in section B will be given greater consideration in assigning a first class mark.
- Casio FX-83 or FX-85 calculators ONLY may be used.
- Students may submit answers to this paper in either Welsh or English.

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Section A

1. Determine which of the following subsets of $\mathbb{R}^3$ are subspaces.

(a) $S_1 = \{(x, y, z) : x + y + z = 1\}$; [3 marks]

(b) $S_2 = \{(x, y, z) : 4x - y - 2z = 0\}$; [5 marks]

(c) $S_3 = \{(x, y, z) : x - y + z \geq 0\}$; [4 marks]

(d) $S_4 = \{(x, y, z) : x^2 - 5yz = 0\}$; [4 marks]

(e) $S_5 = \{(x, y, z) : x^6 + |y| + |z| = 0\}$. [4 marks]

2. Determine which of the following subsets of $\mathbb{R}^3$ are linearly independent.

(a) $\{(0, 6, 7), (3, 2, 1), (0, 0, 0)\}$; [2 marks]

(b) $\{(0, 0, 7), (0, 0, -7), (2, 5, 1)\}$; [2 marks]

(c) $\{(1, 3, 2), (2, 2, 1), (2, -2, 6)\}$; [4 marks]

(d) $\{(1, 5, 2), (4, 2, -1), (2, 4, 1)\}$. [4 marks]

3. Let $\mathcal{P}_3(\mathbb{R})$ be the vector space of polynomials (in the variable x) of degree less than or equal to 3 with real coefficients. Define $\mathcal{F} \subset \mathcal{P}_3(\mathbb{R})$ by

$$\mathcal{F} = \{(2a - b)x^3 + (a + c)x^2 + (b + 2c)x - 6a : a, b, c \in \mathbb{R}\}.$$

Find a spanning set for $\mathcal{F}$. [3 marks]

4. Let $M_{2\times2}(\mathbb{R})$ be the vector space of 2×2 matrices with real entries. Define the subspace $\mathcal{M} \subset M_{2\times2}(\mathbb{R})$ by

$$\mathcal{M} = \text{sp}\left(\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 1 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 2 & 7 \end{pmatrix}\right).$$

Prove that $\dim \mathcal{M} = 2$, where $\dim \mathcal{M}$ denotes the dimension of $\mathcal{M}$. [5 marks]

5. Let $T : \mathcal{P}_3(\mathbb{R}) \to \mathbb{R}^4$ be defined by

$$T(ax^3 + bx^2 + cx + d) = (3a + b, c - d, -6a - 2b, c + 4d),$$

where $a, b, c, d \in \mathbb{R}$.

(a) Prove that T is a linear transformation. [4 marks]

(b) Obtain the matrix representing T with respect to standard bases. [4 marks]

(c) Find the kernel of T and the image space of T, and give bases for both. [4,5 marks]

6. Find all the eigenvalues and eigenspaces of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(x) = Ax$, where

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 4 & 0 & -1 \end{pmatrix}.$$

[9 marks]

Giving appropriate justification, show that A cannot be diagonalised. [4 marks]

SECTION B BEGINS ON THE NEXT PAGE

Section B

7. Let $\mathcal{P}_2(\mathbb{R})$ denote the vector space of polynomials (in the variable x) with degree less than or equal to 2 with real coefficients. Demonstrate that $\{x^2+2x+1, x-1, 3\}$ spans $\mathcal{P}_2(\mathbb{R})$. [4 marks]

8. Let V be a vector space, and let $u, v, w \in V$.

(a) Explain what it means for $\{u, v, w\}$ to be *linearly independent*. [2 marks]

(b) Given that $\{u, v, w\}$ is linearly independent, prove that $\{3u, u+2w, v-w\}$ is linearly independent. [3 marks]

9. Let W_1, W_2 be subspaces of a vector space V.

(a) Prove that:

(i) $W_1 \cap W_2$ is a subspace of V; [3 marks]

(ii) $W_1 + W_2 = \{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}$ is a subspace of V. [3 marks]

(b) Explain what is meant by the *dimension* of a vector space. [2 marks]

(c) Suppose that W_1 and W_2 are finite dimensional. Demonstrate that

$$\text{dim } (W_1 + W_2) + \text{dim } (W_1 \cap W_2) = \text{dim } W_1 + \text{dim } W_2,$$

where dim U denotes the dimension of a vector space U. [10 marks]

(You may use the result that a linearly independent set can be extended to a basis in a finite dimensional space without proof.)

(d) Further suppose V is finite dimensional, and that U is a subspace of V satisfying $V = U + W_1 = U + W_2$ with $U \cap W_1 = U \cap W_2 = \{0\}$. Prove that

$$\text{dim } V - 2\text{dim } U \leq \text{dim } (W_1 \cap W_2) \leq \text{dim } V - \text{dim } U.$$

[4 marks]

10. Let V be a finite dimensional vector space, and let $T : V \to V$ be a linear transformation.

(a) Define the *kernel of* T, the *nullity* of T, the *image space* of T and the *rank* of T, which we denote Ker(T), $\nu(T)$, Im(T) and $\rho(T)$ respectively. [4 marks]

(b) State the dimension theorem for the linear transformation $T : V \to V$. [1 mark]

(c) Prove that

$$\rho(T^2) \leq \frac{1}{2}(\rho(T) + \rho(T^3)),$$

where $T^2, T^3 : V \to V$ denote the linear transformations defined by $T^2(v) = T(T(v))$ and $T^3(v) = T(T(T(v)))$ for $v \in V$ respectively. [10 marks]

11. Let $A \in M_{n\times n}(\mathbb{C})$, the vector space of $n \times n$ matrices with complex entries. Suppose $\lambda, \mu \in \mathbb{C}$ are distinct eigenvalues of A; prove that if $\mathbf{x} \in \mathbb{C}^n$ is an eigenvector corresponding to λ, then $4\mathbf{x}$ is not an eigenvector corresponding to μ. [4 marks]

END OF EXAM PAPER