PRIFYSGOL ABERYSTWYTH UNIVERSITY

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 2 / SEMESTER 2 EXAMINATIONS

MAI – MEHEFIN / MAY – JUNE 2026

MA21510 – Complex Analysis

The questions on this paper are written in English.

Amser a ganiateir - 2 awr

- Gellir rhoi cynnig ar bob cwestiwn.
- Rhoddir mwy o ystyriaeth i berfformiad yn Rhan B wrth bennu marc dosbarth cyntaf.
- Cyfrifianellau Casio FX-83 neu FX-85 YN UNIG a ganiateir.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

Time allowed - 2 hours

- All questions may be attempted.
- Performance in Section B will be given greater consideration in assigning a first class mark.
- Casio FX-83 or FX-85 calculators ONLY may be used.
- Students may submit answers to this paper in either Welsh or English.

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Section A

1. Sketch the portion of the complex plane corresponding to the following inequalities

(a) $Re(z-1) \geq 0$ [3 marks]

(b) $|z+i| \leq 2$ [3 marks]

(c) $|Im(z)| \leq 1$ [4 marks]

2. Find all complex numbers, in the form $z = x+iy$ with $x, y \in \mathbb{R}$ which satisfy $z^3 = -27$. Sketch the solutions on the complex plane. [10 marks]

3. (a) State the Cauchy-Riemann equations for a function f which is analytic. [3 marks]

(b) Show that a function f on $\mathbb{C}$ which is analytic and real-valued must be constant. Hint: Use the Cauchy-Riemann equations to show that $f' = 0$. [7 marks]

4. (a) What condition does a *harmonic function* satisfy? [2 marks]

(b) Verify that $u(x,y) = 2x^2 + x + 1 - 2y^2$ is harmonic (where $x, y \in \mathbb{R}$) [3 marks]

(c) Find a harmonic conjugate $v(x,y)$ of $u(x,y)$, and hence write down an analytic function

$$f(x+iy) = u(x,y) + iv(x,y)$$

satisfying

$$f(0) = 1 + i.$$

[5 marks]

5. (a) Sketch the portion of the complex plane which is the image of the quadrant $\{z = x+iy : x \geq 0, y \geq 0\}$ under the exponential map. [5 marks]

(b) State the definition of the principal branch Log of the logarithm and write

$$Log(1+i\sqrt{3})$$

in the form $x+iy$ with $x, y \in \mathbb{R}$. [5 marks]

6. (a) State Cauchy's integral formula. [2 marks]

(b) If C is the circular contour $|z| = 2$, use Cauchy's integral formula to evaluate

$$\oint_C \frac{\exp \pi z}{(z+i)(z+3i)} dz$$

[8 marks]

7. (a) Compute the Laurent series around $z = 0$ of the function

$$f(z) = \frac{1}{z^4(z+i)}$$

[7 marks]

(b) How can the residue $res_f(0)$ be obtained for the function f of part (a) from the Laurent series? Compute $res_f(0)$ explicitly. [3 marks]

Section B

8. Evaluate the following integrals

(a)

$$\int_{C_1} |z|^2 dz$$

where C_1 is the straight line segment from $1+i$ to $2-i$. [8 marks]

(b)

$$\int_{C_2} e^z dz$$

where C_2 is the half circle from π through $i\pi$ to $-\pi$. [2 marks]

9. (a) State the definition of a simple pole. [1 mark]

(b) State the limit formula for the evaluation of a residue at a simple pole. [1 mark]

(c) Suppose p and q are polynomials such that $p(0) \neq 0$ and $q(0) = 0$ and $q'(0) \neq 0$. Show that 0 is a simple pole for the rational function $f(z) = p(z)/q(z)$ and prove that in this case the residue of f at 0 can be computed as

$$res_f(0) = \frac{p(0)}{q'(0)}.$$

[8 marks]

10. Use the Residue Theorem to evaluate the following integrals:

(a)

$$\oint_{|z|=3} \frac{e^{2z}}{z^2+4z+4} dz,$$

[7 marks]

(b)

$$\oint_{|z+1-i|=1} \frac{1}{z^4+4} dz,$$

[10 marks]

(c)

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+2)} dx,$$

[8 marks]

(d)

$$\int_0^{2\pi} (\sin t)^6 dt.$$

[5 marks]