PRIFYSGOL ABERYSTWYTH UNIVERSITY

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 2 / SEMESTER 2 EXAMINATIONS

MAI – MEHEFIN / MAY – JUNE 2026

MA25220 – Introduction to Numerical Analysis and its Applications

The questions on this paper are written in English.

Amser a ganiateir - 2 awr

- Gellir rhoi cynnig ar bob cwestiwn.
- Rhoddir mwy o ystyriaeth i berfformiad yn Rhan B wrth bennu marc dosbarth cyntaf.
- Cyfrifianellau Casio FX-83 neu FX-85 YN UNIG a ganiateir.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

Time allowed - 2 hours

- All questions may be attempted.
- Performance in Section B will be given greater consideration in assigning a first class mark.
- Casio FX-83 or FX-85 calculators ONLY may be used.
- Students may submit answers to this paper in either Welsh or English.

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Section A

1. For the following data:

i	0	1	2
x_i	-1	0	1
$f(x_i)$	2	1	-1

use Lagrange's method to

(a) construct the linear spline; [6 marks]

(b) find the quadratic interpolating polynomial. [5 marks]

2. (a) Construct the divided difference table for the following data:

x	-2	0	1	2	5
$f(x)$	-15	1	-3	-7	41

[6 marks]

(b) State the lowest degree of polynomial which matches the data exactly. [1 mark]

(c) Determine the interpolating polynomial using Newton's divided difference formula

$$p_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, .., x_k] \prod_{i=0}^{k-1} (x - x_i).$$

Give the polynomial in its simplest form. [4 marks]

3. A function f is given in tabulated form below.

x	1	1.25	1.5	1.75	2
$f(x)$	10	8	7	6	5

Use the trapezoidal rule to approximate $I(f) = \int_1^2 f(x)\,dx$,

(a) using only the function values at $x = 1$ and 2; [2 marks]

(b) using all of the given function values. [4 marks]

4. The error formula for the composite Simpson approximation of

$$I(f) = \int_a^b f(x)dx,$$

using $2n$ subintervals with width $h = (b-a)/2n$, is

$$E_{S_{2n}}(f) = I(f) - S_{2n} = -\frac{(b-a)h^4}{180} f^{(4)}(\xi), \quad \text{for some } \xi \in (a,b).$$

Find the number of subintervals required so that the composite Simpson rule is guaranteed to estimate

$$I = \int_0^2 \cos(3x)\, dx$$

correct to 5 decimal places. [8 marks]

5. (a) Show that the function

$$f(x) = x^3 + 3x - 5$$

has a root in the interval $[1,2]$. [2 marks]

(b) Use the bisection method to find this root to a tolerance $\varepsilon = 0.05$. [6 marks]

(c) Show that the bisection method is a first order method for computing the root. [3 marks]

6. (a) Show that $g(x) = \sqrt[3]{x+1}$ is a contraction mapping on $[1,2]$. [6 marks]

(b) Use the fixed point method

$$x_{r+1} = \sqrt[3]{x_r + 1},$$

to find the root of the equation

$$x^3 - x - 1 = 0$$

in $[1,2]$, correct to 4 decimal places. [5 marks]

7. Consider the initial value problem

$$y' = 1 - y - e^{-x},\ y(0) = 1.$$

(a) Show that the function $f(x, y) = 1 - y - e^{-x}$, satisfies a Lipschitz condition with respect to y, so that the problem has a unique solution for $0 \leq x \leq 1$. [4 marks]

(b) Show that Euler's method is consistent with the initial value problem. That is, show that

$$\lim_{h \to 0} T(x; h) = 0,$$

where $T(x; h)$ denotes the local truncation error. [2 marks]

(c) Use Euler's method with $h = 0.25$ to estimate the value of the solution for $0 \leq x \leq 1$. [6 marks]

Section B

8. Construct the cubic spline $S_3(x)$ for the data

x	0	1	2
y	1	0	3

using the natural boundary conditions $S_3''(0) = S_3''(2) = 0$. [14 marks]

9. Determine the values of w_1, x_0 and x_1 to obtain the Gaussian quadrature rule

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\,dx = \frac{\pi}{2} f\left(x_0\right) + w_1 f\left(x_1\right) + E,$$

where E is zero when $f(x)$ is a polynomial of degree 2 or less.
Hint: You may use the standard integral

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C,$$

without proof. [14 marks]

10. Recall that Newton's method for finding a root, x^*, of a non-linear equation $f(x) = 0$ is given by

$$x_{r+1} = x_r - \frac{f\left(x_r\right)}{f'\left(x_r\right)},$$

for $r = 0, 1, \ldots$.
Show that this method exhibits quadratic convergence provided that $f'\left(x^*\right) \neq 0$. [8 marks]

11. (a) Use the predictor-corrector pair

$$y_{n+1}^{[p]} = y_n + h f\left(x_n, y_n\right),$$

$$y_{n+1}^{[c]} = y_n + \frac{h}{2}\left(f\left(x_n, y_n\right) + f\left(x_{n+1}, y_{n+1}^{[p]}\right)\right),$$

to estimate the solution of the initial value problem

$$y' = x^2 - y,\ 0 \leq x \leq 1,\ y(0) = 0.5,$$

with $h = 0.25$. [7 marks]

(b) What restriction on h, if any, is required for this predictor-corrector method to be absolutely stable? [7 marks]