PRIFYSGOL ABERYSTWYTH UNIVERSITY

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 2 / SEMESTER 2 EXAMINATIONS

MAI / MAY 2026

MA25610 – Hydrodynamics I

The questions on this paper are written in English.

Amser a ganiateir - 2 awr

- Gellir rhoi cynnig ar bob cwestiwn.
- Rhoddir mwy o ystyriaeth i berfformiad yn Rhan B wrth bennu marc dosbarth cyntaf.
- Cyfrifianellau Casio FX-83 neu FX-85 YN UNIG a ganiateir.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

Time allowed - 2 hours

- All questions may be attempted.
- Performance in Section B will be given greater consideration in assigning a first class mark.
- Casio FX-83 or FX-85 calculators ONLY may be used.
- Students may submit answers to this paper in either Welsh or English.

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Useful formulae
In the usual notation:

$$\nabla\phi = \frac{1}{h_1}\frac{\partial\phi}{\partial x_1}\underline{e}_1 + \frac{1}{h_2}\frac{\partial\phi}{\partial x_2}\underline{e}_2 + \frac{1}{h_3}\frac{\partial\phi}{\partial x_3}\underline{e}_3$$

$$\nabla.\underline{u} = \frac{1}{h_1h_2h_3}\left(\frac{\partial(h_2h_3u_1)}{\partial x_1} + \frac{\partial(h_1h_3u_2)}{\partial x_2} + \frac{\partial(h_1h_2u_3)}{\partial x_3}\right)$$

$$\nabla \wedge \underline{u} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\underline{e}_1 & h_2\underline{e}_2 & h_3\underline{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ h_1u_1 & h_2u_2 & h_3u_3 \end{vmatrix}$$

$$\nabla^2\phi = \frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial x_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\phi}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial x_3}\right)\right)$$

Euler's equations relating velocity $\underline{u}$ and pressure p are

$$\rho\frac{D\underline{u}}{Dt} = -\underline{\nabla}p + \underline{F}.$$

In plane polar coordinates (r, θ), without body forces, the r and θ components are

$$\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta^2}{r}\right) = -\frac{\partial p}{\partial r}$$

and

$$\rho\left(\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_ru_\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial\theta}$$

respectively.

Section A

1. (a) Define the Reynolds number for fluid flow, carefully explaining your notation. [4 marks]

(b) Estimate the Reynolds number associated with water waves moving towards Aberystwyth sea-front in winter. You may assume that the physical properties of sea-water are $\rho \approx 10^3 \mathrm{kg/m}^3$ and $\mu \approx 10^{-3}$ Pa s. [3 marks]

(c) Explain whether or not an inviscid fluid would provide a good model for this flow. [2 marks]

2. Consider the steady two-dimensional velocity fields given below in either Cartesian (x, y) or plane polar (r, θ) coordinates, where λ is a positive constant. In each case determine an equation for the pathlines, sketch them on Cartesian axes, and indicate the direction of fluid motion.

(a) $\underline{u} = \lambda x \underline{e}_x - \lambda y \underline{e}_y$, [6 marks]

(b) $\underline{u} = \lambda r \underline{e}_\theta$. [6 marks]

3. (a) Give an expression for the convective derivative $\dfrac{\mathrm{D}\underline{u}}{\mathrm{D}t}$ of a vector field $\underline{u}$. [3 marks]

(b) Calculate the convective derivative of the velocity field

$$\underline{u} = 3xt\underline{e}_x - 3yzt\underline{e}_y + z\underline{e}_z.$$

[6 marks]

4. (a) Calculate the vorticity $\underline{\omega}$ of the following fluid flows:

(i) $\underline{u} = x\underline{e}_y$, [3 marks]

(ii) $\underline{u} = \dfrac{a}{r}\underline{e}_r$, [3 marks]

where a is a positive constant.

(b) State which, if any, of these flows are irrotational. [1 mark]

. . . questions continue

5. Determine whether or not the following flows are suitable for incompressible fluids:

(a) $\underline{u} = x\underline{e}_x - y\underline{e}_y$, [3 marks]

(b) $\underline{u} = \dfrac{a}{r}\underline{e}_r$, [3 marks]

where a is a positive constant.

6. (a) Define what it means for a velocity field $\underline{u} = u\underline{e}_x + v\underline{e}_y$ to have a stagnation point. [2 marks]

(b) Determine the position of the stagnation point(s) in a flow with velocity field $\underline{u} = ay\underline{e}_x + (x-a)\underline{e}_y - z\underline{e}_z$, where a is a positive constant. [4 marks]

7. (a) State Bernoulli's equation for a fluid with pressure p, density ρ and velocity $\underline{u}$ subject to a body force with potential χ, and the conditions under which it holds. [7 marks]

(b) Petrol flows from a tap of diameter 1cm, through a pipe, and then out of a nozzle with diameter 5mm. The petrol can be assumed to be incompressible. If the petrol leaves the tap with a speed of $u_{tap} = 1$ m/s, at what speed does it flow out of the nozzle? [3 marks]

(c) Taking the acceleration due to gravity to be $g = 10\mathrm{m/s}^2$, if the pipe descends by hm over its length, what is the change in the pressure in the petrol between the tap and the nozzle in terms of h and the density ρ of the petrol? [3 marks]

8. Consider the flow $\underline{u} = 2xy\underline{e}_x + (x^2 - y^2)\underline{e}_y$.

(a) Calculate the velocity potential ϕ for this flow. [3 marks]

(b) Calculate the stream function ψ for this flow. [3 marks]

(c) Show that the velocity potential that you have found is harmonic, i.e. that it satisfies Laplace's equation. [2 marks]

. . . questions continue

Section B

9. Consider the unsteady two-dimensional velocity field $\underline{u} = t\underline{e}_x + \dfrac{3}{y}\underline{e}_y$, where t is time.

(a) Give an equation $y(x)$ for the streamline through the point (x_0, y_0). [5 marks]

(b) Show that the pathline through the point (x_0, y_0) has equation

$$72(x - x_0) = \left(y^2 - y_0^2\right)^2.$$

[5 marks]

10. Consider the following boundary value problem for the velocity potential ϕ for a two-dimensional fluid flow:

$$\nabla^2\phi = 0 \text{ on } r > a; \quad \left.\frac{\partial\phi}{\partial r}\right|_{r=a} = 0; \quad \lim_{r\to\infty}\nabla\phi = U\underline{e}_x,$$

where U and a are positive constants.

(a) Describe this flow. [2 marks]

(b) Show that the far-field boundary condition can be written as $\phi \to Ur\cos\theta$ as $r \to \infty$. [2 marks]

(c) Find a separable solution of Laplace's equation in plane polar coordinates (r, θ), apply both boundary conditions, and hence find the solution for ϕ. [11 marks]

11. A fluid flows with velocity field $\underline{u} = (x^2 - y^2)\underline{e}_x - 2xy\underline{e}_y$. If the pressure in the fluid is p_0 at the origin, determine how the pressure varies throughout the fluid in the absence of body forces. [11 marks]

12. Calculate the convective derivative of the velocity field

$$\underline{u} = \cos\theta\underline{e}_r + 2r^2\theta\underline{e}_\theta + rz\underline{e}_z,$$

written in cylindrical polar coordinates. [4 marks]

. . . questions continue

13. An homogeneous inviscid liquid extends to infinity where the pressure is zero. In it is a spherical bubble of gas in which the pressure p (assumed to be constant throughout the bubble) is inversely proportional to the bubble volume to the power of $4/3$. The fluid is initially at rest, with the radius of the bubble equal to a and the pressure in the bubble equal to p_0.

Let $R(t)$ be the radius of the bubble, which satisfies

$$\frac{R^2\ddot{R}}{r} + \frac{2R\dot{R}^2}{r} - \frac{R^4\dot{R}^2}{2r^4} = \frac{p}{\rho},$$

where p and ρ are the pressure and density of the liquid respectively, and r is radial position in the liquid outside the bubble.

(a) Show that the pressure p in the bubble satisfies

$$p = p_0\left(\frac{a}{R}\right)^4.$$

[4 marks]

(b) Use the identity

$$2R^2\left(R\ddot{R} + \frac{3}{2}\dot{R}^2\right) = \frac{\mathrm{d}}{\mathrm{d}R}\left(R^3\dot{R}^2\right),$$

to show that in the subsequent motion the rate of growth of the bubble satisfies the equation

$$\rho\dot{R}^2 = 2p_0\left(\frac{a^3}{R^3} - \frac{a^4}{R^4}\right).$$

[6 marks]

End of exam paper