PRIFYSGOL ABERYSTWYTH UNIVERSITY

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 2 / SEMESTER 2 EXAMINATIONS

MAI – MEHEFIN / MAY – JUNE 2026

MA25710 – Advanced Dynamics

The questions on this paper are written in English.

Amser a ganiateir - 2 awr

- Gellir rhoi cynnig ar bob cwestiwn.
- Rhoddir mwy o ystyriaeth i berfformiad yn Rhan B wrth bennu marc dosbarth cyntaf.
- Cyfrifianellau Casio FX-83 neu FX-85 YN UNIG a ganiateir.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

Time allowed - 2 hours

- All questions may be attempted.
- Performance in Section B will be given greater consideration in assigning a first class mark.
- Casio FX-83 or FX-85 calculators ONLY may be used.
- Students may submit answers to this paper in either Welsh or English.

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Section A

1. (a) A satellite travels in a circular orbit around the earth. The **speed** S of the satellite depends on the **gravitational constant** G, the **mass** M of the earth and **radius** r of the orbit.

(i) Using dimensional analysis find an expression for the speed of the satellite. Note the gravitational constant has dimensions of $[\text{length}]^3[\text{mass}]^{-1}[\text{time}]^{-2}$. [8 marks]

(ii) What happens to the speed of the satellite if the radius of the orbit is increased by a factor of four? [2 marks]

(b) The **frequency** F of sound produced by a tuning fork depends on its **length** L as well as the **elastic constant** Y and **density** ρ of the material from which it is made.

(i) Using dimensional analysis find an expression for the frequency of the tuning fork. Note the elastic constant has dimensions of $[\text{length}]^{-1}[\text{mass}][\text{time}]^{-2}$. [8 marks]

(ii) What happens to the frequency of the tuning fork if its length is doubled *and* its density is decreased by a factor of four? [2 marks]

2. Find the extremal $y(x)$ for the following functional

$$I = \int_{x_1}^{x_2} \left(e^x y(x) + 4[y'(x)]^2 \right) dx,$$

where $y'(x) = \dfrac{dy(x)}{dx}$. [10 marks]

3. (a) Show that the Euler-Lagrange equation for the functional

$$I = \int_0^X \left([y'(x)]^2 - [y(x)]^2 \right) \, dx, \quad \text{with} \quad y(0) = 0, \quad y(X) = 1,$$

with $0 < X < \pi$ is given by

$$y''(x) + y(x) = 0,$$

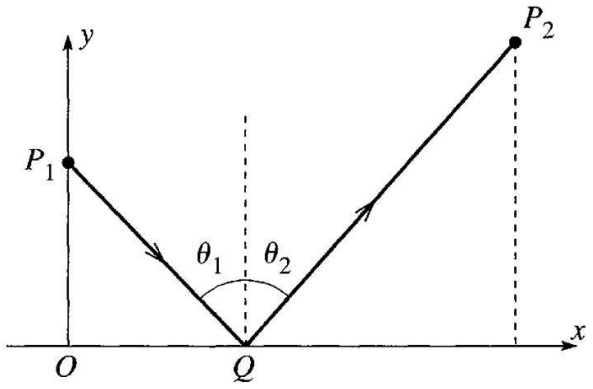
where $y' = \dfrac{dy(x)}{dx}$ and $y'' = \dfrac{d^2y(x)}{dx^2}$. [5 marks]

(b) Hence, show that the stationary function is

$$y = \frac{\sin x}{\sin X},$$

where $0 < X < \pi$. [5 marks]

4. Consider a ray of light traveling in a vacuum. The light ray originates at the point $P_1 = (0, y_1, 0)$, is then reflected by a plane mirror at the point $Q = (x, 0, z)$ and arrives at the point $P_2 = (x_2, y_2, 0)$, as in the figure below - note, the x and y directions are as show in the diagram while the z direction is into the page.



(a) Show that the total time taken for the light beam to travel from P_1 to Q and then from Q to P_2 is given by

$$T = \frac{1}{c}\sqrt{x^2 + y_1^2 + z^2} + \frac{1}{c}\sqrt{(x_2 - x)^2 + y_2^2 + z^2},$$

where c is the speed of light in a vacuum. [2 marks]

(b) Hence, using Fermat's principle demonstrate that

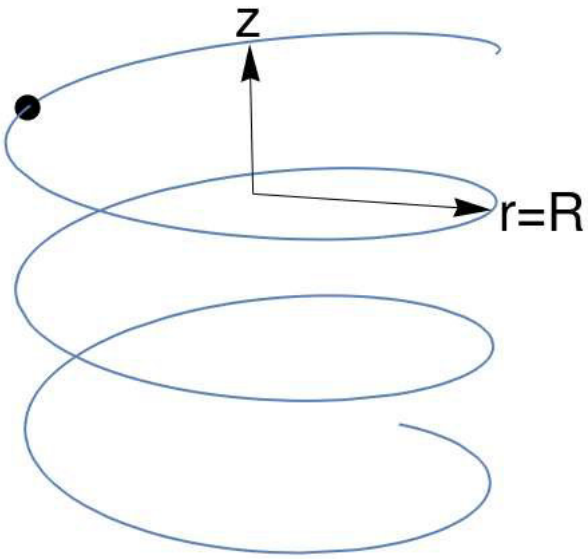
(i) the path taken by the light beam is in the $z = 0$ plane,

(ii) and for the light beam $\theta_1 = \theta_2$.

Here θ_1 is the incident angle (i.e. the angle between the incident ray and the normal) and θ_2 is the reflected angle (i.e. the angle between the reflected ray and the normal) - as shown on the diagram.

[8 marks]

5. A bead of mass m is slipped onto a frictionless wire wound in the shape of a helix of radius R. The helix can be described using cylindrical polar coordinates $\mathbf{r} = (r = R, \theta, z)$, where for any given point on the helix the angle θ is related to the height z by $\theta = \alpha z$, where α is a constant.



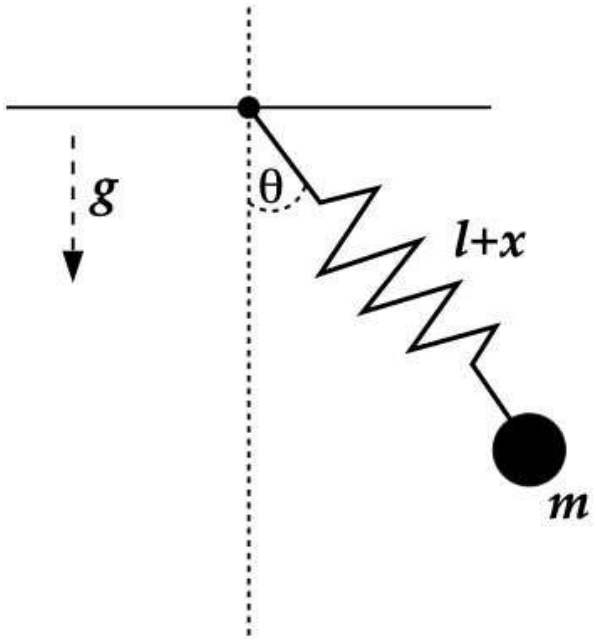
(a) Suppose that the mass slides down the helix under the action of gravity g. Show that the Lagrangian for the mass is given by

$$L = \frac{1}{2}m(\alpha^2 R^2 + 1)\dot{z}^2 - mgz,$$

where $\dot{z} = \dfrac{dz(t)}{dt}$. [6 marks]

(b) Hence, find an expression for the angular velocity of the mass as it slides down the helix. [4 marks]

6. Consider a pendulum composed of a spring, with spring constant k, attached to a mass m, as shown in the diagram. The spring is arranged to lie in a plane and gravity g acts down the plane as shown in the figure.

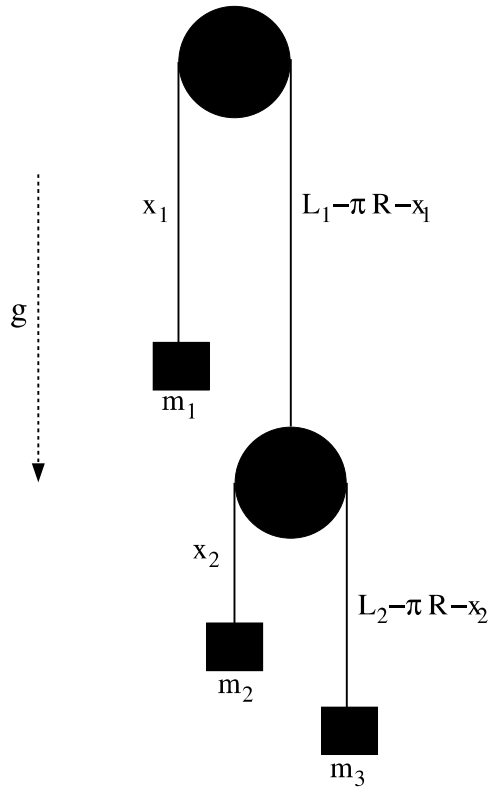


Suppose that the equilibrium length of the spring is l and at any time t the spring has a length $l + x(t)$. Assuming that $\theta(t)$ is the angle the pendulum makes with the vertical, find the equations of motion of the mass. [10 marks]

Section B

7. Consider the dynamical system shown in the figure below. A mass m_1 is connected by a light inextensible cord of length L_1 over a pulley (which is fixed in place) to a second pulley (of negligible mass). The second pulley connects a mass m_2 by a light inextensible cord of length L_2 to a mass m_3.

Assume that both pulleys have a radius R and that gravity acts downwards (as shown in the figure).



(a) Show that the Lagrangian for the dynamical system is given by

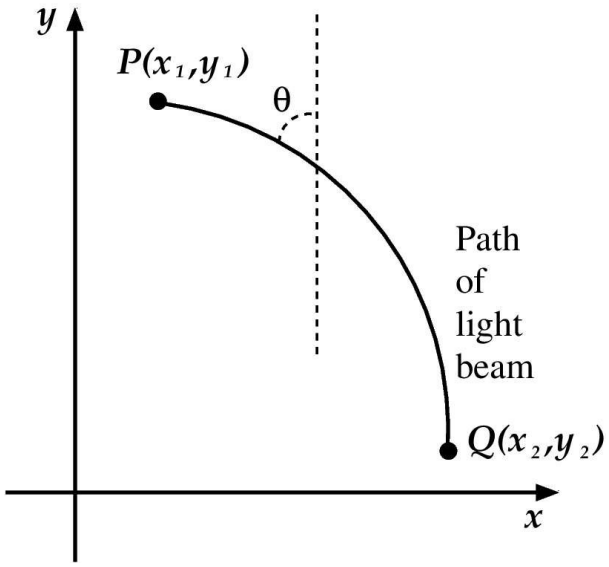
$$\begin{aligned} L &= \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2(-\dot{x}_1+\dot{x}_2)^2 + \frac{1}{2}m_3(-\dot{x}_1-\dot{x}_2)^2 \\ &+ g(m_1-m_2-m_3)x_1 + g(m_2-m_3)x_2 + C \end{aligned}$$

where C is a constant, and $\dot{x}_1 = \dfrac{dx_1(t)}{dt}$ & $\dot{x}_2 = \dfrac{dx_2(t)}{dt}$. [7 marks]

(b) Hence, using the Lagrangian from the previous part find the equations of motion for the system. [8 marks]

8. As a ray of light from a distant star descends through the atmosphere of the Earth it travels through air which has an increasing refractive index.

The Earth is essentially flat over the horizontal range of the ray, and is represented by the x-axis in the figure below. Take the refractive index to be given by $n = n(y)$, where y is the height above the earth and let θ be the angle the path of the light beam makes with the vertical axis at a given point in time



(a) Show that the time taken for the light beam to travel between the two points P and Q, as shown in the diagram, is given by

$$T = \frac{1}{c}\int_{y_1}^{y_2} n(y)\sqrt{1+x'^2}\,dy,$$

where c is the speed of light in a vacuum. [4 marks]

(b) Hence, show that the path followed by the light beam is given by the equation

$$\frac{n(y)x'}{\sqrt{1-x'^2}} = k,$$

where k is a constant and $x' = \dfrac{dx(y)}{dy}$. [4 marks]

(c) Using your answer from the previous part show that

$$\theta = \sin^{-1}\left[\frac{k}{n(y)}\right].$$

[4 marks]

(d) Suppose the refractive index varies slowly with height according to

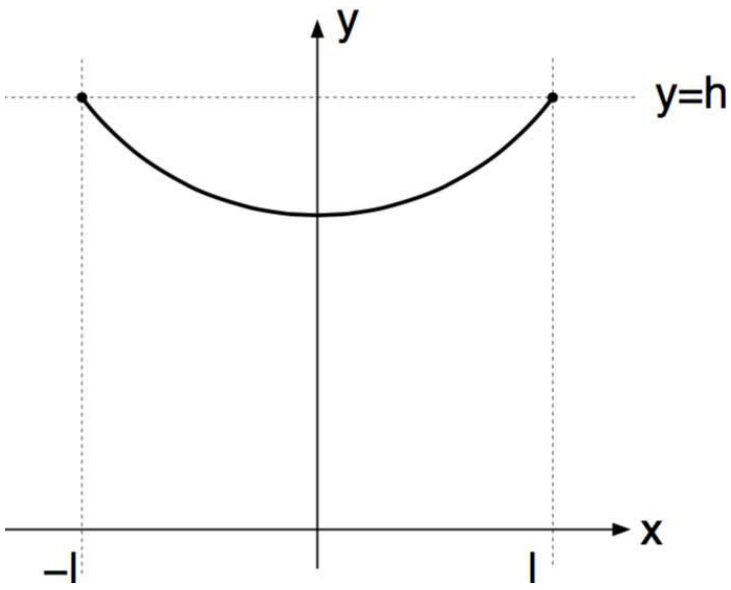
$$n(y) = n_0(1-\alpha y), \qquad \alpha y \ll 1,$$

where n_0 and α are positive constants. Show that the angle θ satisfies

$$\sin\theta \approx \frac{k}{n_0}(1+\alpha y).$$

[2 marks]

9. Consider a rope of length L hanging between the two points (l, h) and $(-l, h)$, with $h > 0$ as illustrated in the figure. Assume the length L satisfies the inequality $L > 2l$.



Let ρ denote the mass density per unit length of the rope and assume this density is a constant along the length of the rope.

(a) Show that the potential energy of the rope is given by,

$$V = \rho g \int_{-l}^{l} y\sqrt{1 + y'^2}dx,$$

where $y' = \frac{dy}{dx}$. [3 marks]

(b) Hence, show that the shape of the hanging rope is given by the extremal of the integral

$$I = \rho g \int_{-l}^{l} F(x, y, y')dx,$$

where,

$$F(x, y, y') = y\sqrt{1 + y'^2} + \lambda^*\sqrt{1 + y'^2},$$

$\lambda^* = \frac{\lambda}{\rho g}$, and λ is the Lagrange multiplier. [4 marks]

(c) By solving the Euler-Lagrange equation for $F(x, y, y')$ show that

$$C = \frac{y + \lambda^*}{\sqrt{1 + y'^2}},$$

where C is a constant. [4 marks]

(d) Show that the result from part (c) can be written as

$$\frac{dy}{\sqrt{(y+\lambda^*)^2 - C^2}} = \pm\frac{dx}{C}.$$

[3 marks]

(e) By making the substitution $y + \lambda^* = C\cosh\theta$ in the equation from the previous part show that the shape of the hanging rope is given by

$$y(x) = C\cosh\left(\frac{x+k}{C}\right) - \frac{\lambda}{\rho g},$$

where k is a constant.

Note that you may require the following identities:

$$\cosh^2 u - \sinh^2 u = 1,$$

and

$$\frac{d}{du}\cosh u = \sinh u.$$

[7 marks]