PRIFYSGOL ABERYSTWYTH UNIVERSITY

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 2 / SEMESTER 2 EXAMINATIONS

MAI – MEHEFIN / MAY – JUNE 2026

MA30110 – Group Theory

The questions on this paper are written in English.

Amser a ganiateir - 2 awr **Time allowed - 2 hours**

- Gellir rhoi cynnig ar bob cwestiwn.
- Rhoddir mwy o ystyriaeth i berfformiad yn rhan B wrth bennu marc dosbarth cyntaf.
- Cyfrifianellau Casio FX-83 neu FX-85 YN UNIG a ganiateir.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

- All questions may be attempted.
- Performance in section B will be given greater consideration in assigning a first class mark.
- Casio FX-83 or FX-85 calculators ONLY may be used.
- Students may submit answers to this paper in either Welsh or English.

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Section A

1. Express the permutation

$$\pi = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 8 & 1 & 3 & 7 & 5 & 6 & 2 \end{bmatrix}$$

(a) as a product of disjoint cycles, [3]

(b) as a product of transpositions. [3]

(c) What is the order of π ? [3]

(d) Determine whether π is even or odd. [3]

(e) Find an element which is different from π but conjugate to π. [3]

(f) In the symmetric group $S(8)$, find an odd element of order 6 and an even element of order 6. [4]

(19 marks)

2. In the symmetric group $S(5)$ let H be the subgroup generated by the cycle (123). How many distinct left cosets of H are there? Choose one of these left cosets which is different from H and compute all its elements in cycle notation. (3+6=9 marks)

3. (a) What does Lagrange's theorem tell us about the order of a subgroup of a finite group?

Suppose that in a group G of order 240 you find an element a of order 24 and an element b of order 10.

(b) What is the definition of the subgroup $\langle a, b\rangle$ generated by a and b?

(c) Use Lagrange's theorem to determine the order of $\langle a, b\rangle$, assuming that $\langle a, b\rangle \neq G$. Explain your reasoning.

(2+2+4= 8 marks)

4. In the cyclic group C_{30} let g be a generator, so that you can write all elements as powers of g.

(a) Write down an explicit list of all subgroups of C_{30} where each subgroup appears on the list exactly once.

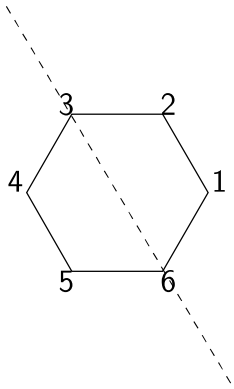
(b) Identify in your list the subgroup generated by g^7 and the subgroup generated by g^8. Justify your choice.

(5+5=10 marks)

5. Let H be a subgroup of a group G.

(a) What does it mean for H to be normal? [2]

(b) Let G be the symmetry group of a regular hexagon (as drawn below) and let H be the subgroup generated by the reflection r on the dashed line from 3 to 6. Show that this subgroup is not normal. [5]



(c) State *Burnside's Lemma*. [3]

(d) The symmetry group G of the regular hexagon acts on the vertices $1, \ldots, 6$ (as drawn above). Verify Burnside's Lemma for this action. [4]

(e) Now we colour the 6 vertices of the regular hexagon with black or white. A colouring is defined as the choice of one of the colours for each vertex. The symmetry group G of the regular hexagon acts on the set of colourings by rotations and reflections. Use Burnside's Lemma to count the orbits of this action. [6]

(f) Now suppose that instead of the full symmetry group we only consider the action of the subgroup K of rotations on the colourings. Count the number of orbits for this action. [4]

(24 marks)

Section B

6. (a) For a group action state the definitions of stabiliser and orbit and the Orbit-Stabiliser Theorem.

We consider the action of the symmetric group $S(5)$ on itself by conjugation.

(b) How many elements are in the orbit of the cycle (12345) ?

(c) From (b) and the Orbit-Stabiliser Theorem compute the number of elements in the stabiliser of the cycle (12345).

(d) Find all elements in the stabiliser of the cycle (12345).

(3+3+2+3=11 marks)

7. (a) Prove that subgroups of a cyclic group are always cyclic.

(b) Prove that a group with prime order p is always cyclic.

(c) Prove that in a group G of order n the n-th power of every element is equal to the identity.

(d) Prove that for a group G of order $n \geq 2$ the direct product $G \times G$ is not cyclic.

(5+4+3+5=17 marks)

8. Find a non-trivial homomorphism from the cyclic group C_{100} to the cyclic group C_{14} (by 'non-trivial' we mean here that its image includes not only the identity element). Show that it is well-defined and determine its kernel and its image. (8 marks)

9. Suppose that G is a group of order 198.

(a) Show that G has a non-trivial normal subgroup. State explicitly the Sylow theorems which you use for your argument.

(b) Suppose further that G is abelian but not cyclic. Identify G as a direct product of cyclic groups. Explain that, given the assumptions, this is unique up to isomorphism.

(7+7=14 marks)