PRIFYSGOL ABERYSTWYTH UNIVERSITY

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 2 / SEMESTER 2 EXAMINATIONS

MAI / MAY 2026

MA34210 – Asymptotic Methods in Mechanics

The questions on this paper are written in English.

Amser a ganiateir - 2 awr

- Gellir rhoi cynnig ar bob cwestiwn.
- Rhoddir mwy o ystyriaeth i berfformiad yn Rhan B wrth bennu marc dosbarth cyntaf.
- Cyfrifianellau Casio FX-83 neu FX-85 YN UNIG a ganiateir.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

Time allowed - 2 hours

- All questions may be attempted.
- Performance in Section B will be given greater consideration in assigning a first class mark.
- Casio FX-83 or FX-85 calculators ONLY may be used.
- Students may submit answers to this paper in either Welsh or English.

Formulae:
You may find the following Taylor-Maclaurin series and definite integral useful:

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}, \quad \text{converges for } x \in \mathbb{R}.$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \text{converges for } x \in \mathbb{R}.$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad \text{converges for } |x| < 1.$$

$$\int_{-\infty}^{\infty} e^{-x^2} \mathrm{d}x = \sqrt{\pi}.$$

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Section A

1. Consider the following cubic equation, where ε is a small parameter:

$$x^3 - x^2 - 6x - 3\varepsilon = 0,$$

(a) Find all roots of the limit problem corresponding to $\varepsilon = 0$. [4 marks]

(b) Clearly explaining your reasoning, state whether the problem is a *regular* or *singular* perturbation of the limit problem corresponding to $\varepsilon = 0$. [3 marks]

(c) Obtain two-term asymptotic approximations (in the form $x_0 + \varepsilon x_1 + O(\varepsilon^2)$ as $\varepsilon \to 0$) of all three roots of the cubic equation. [10 marks]

2. Consider the following function $f : (0, 1) \to \mathbb{R}$, which is defined by an integral:

$$f(x) = \int_0^x \frac{\cos(2\beta)}{1 - \sin\beta} \mathrm{d}\beta, \qquad 0 < x < 1.$$

(a) Obtain an asymptotic expansion for $f(x)$ as $x \to 0$, giving your answer in the form

$$f(x) = b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + O(x^5), \quad x \to 0,$$

where b_1, b_2, b_3, and b_4 are constants you should determine. [8 marks]

(b) Hence or otherwise, state whether each of the following asymptotic statements is true or false (you are not required to justify your assertions):

(i) $f(x) \sim x^2 \cos x, \quad x \to 0$;
(ii) $f(x) = O(x), \quad x \to 0$;
(iii) $f(x) = o(x), \quad x \to 0$. [3 marks]

3. Consider the following initial value problem for a regularly perturbed ordinary differential equation (ODE), in which $\varepsilon > 0$ is a small parameter:

$$\begin{cases} \dfrac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \varepsilon \dfrac{\mathrm{d}y}{\mathrm{d}t} = -1, \quad t > 0; \\ y(0) = 0; \\ y'(0) = 1. \end{cases} \tag{1}$$

(a) Classify the ODE as either linear or non-linear, and state its order. [2 marks]

(b) Obtain a three term asymptotic approximation of the initial value problem (1), i.e. obtain an approximation of the form

$$y_a(t) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + O(\varepsilon^3), \ \varepsilon \to 0.$$

[11 marks]

4. This question concerns the following singularly perturbed quartic equation:

$$\varepsilon^4 x^4 + \varepsilon^2 x^3 - x^2 + x + 1 = 0.$$

(a) State how many of this quartic equation's roots in $\mathbb{C}$ are regularly perturbed and how many are singularly perturbed. [2 marks]

(b) Use the singular perturbation ansatz

$$x(\varepsilon) = \varepsilon^{-2} z(\varepsilon), \quad z(0) \neq 0,$$

to find a leading order (i.e. one-term) approximation of the singularly perturbed root that diverges to $+\infty$ as $\varepsilon \to 0$. Hence give a numerical approximation of this root when $\varepsilon = 0.01$. [8 marks]

5. (a) Give the formal definition of what is meant by the statement: $f(x) = O(g(x))$ as $x \to 0$. [2 marks]

(b) Prove that $3xe^x \cos^2(x) = O(x)$ as $x \to 0$. [3 marks]

(c) Give the formal definition of what is meant by the statement: $f(x) = o(g(x))$ as $x \to \infty$. [2 marks]

(d) Prove that the following is an *invalid* asymptotic estimate:

$$7x + 8 = o(x) \text{ as } x \to \infty.$$

[4 marks]

(e) Give the definition of the statement: $f(x) \sim g(x)$ as $x \to 0$. [2 marks]

(f) Let $a, b, c : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be defined as:

$$a(x) = x^{-2}\cos(2x); \qquad b(x) = x^{-4}\sin(3x); \qquad c(x) = e^x \cos(5x).$$

For each of a, b, c, state the values of k and n for which the function is asymptotically equivalent to kx^n as $x \to 0$. [6 marks]

Section B

6. Draw a suitable Kruskal-Newton graph and use appropriate asymptotic ansatzes to find two-term approximations of all three roots of the singularly perturbed cubic equation

$$\varepsilon^3 x^3 - \varepsilon x^2 - x + 2 = 0.$$

[14 marks]

7. The van der Pol equation with small non-linear damping,

$$y''(t) + \varepsilon(y^2(t) - 1)y'(t) + y(t) = 0,$$

describes a self-excited oscillator whose energy increases when $|y| < 1$ and decreases when $|y| > 1$.

(a) Use regular perturbation methods to find a two-term approximation to a solution of the van der Pol equation with initial conditions $y(0) = a_0$, and $y'(0) = 0$, where $a_0 > 0$. The hint stated at the end of this question may be useful. [9 marks]

(b) State what secular terms are, identify any such terms in your two-term approximation from (a), and briefly outline why secular terms are undesirable when modelling an oscillator. [3 marks]

(c) Use the Lindstedt-Poincaré method to find a two-term asymptotic approximation of a non-trivial *periodic* solution of the van der Pol equation with the same initial conditions as described in (a). State clearly the frequency and value of a_0 required to obtain periodic oscillations. Again, the hint may be of use.

[15 marks]

Hint: you may find it useful to note that the general solution to

$$y''(t) + y(t) = A\cos^2(t)\sin(t) - B\sin(t) + C\cos(t)$$

is given by

$$y(t) = c_1\cos t + c_2\sin t + \frac{1}{16}\left((8B - 2A)t\cos t - A\cos(2t)\sin t + 8Ct\sin t\right),$$

where c_1, c_2 are arbitrary constants.

8. Clearly explaining each step in your working, use Laplace's method to determine constants α, β, γ such that

$$\int_{-1}^{1} \cos(x^2)e^{-\lambda(3+\sin^2 x)}\mathrm{d}x \sim \alpha\lambda^\beta e^{\gamma\lambda}, \qquad \lambda \to +\infty.$$

[9 marks]