PRIFYSGOL ABERYSTWYTH UNIVERSITY

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 2 / SEMESTER 2 EXAMINATIONS

MAI / MAY 2026

MA34920 – Mathematical Models of Biological Systems

The questions on this paper are written in English.

Amser a ganiateir - 2 awr

- Gellir rhoi cynnig ar bob cwestiwn.
- Rhoddir mwy o ystyriaeth i berfformiad yn Rhan B wrth bennu marc dosbarth cyntaf.
- Cyfrifianellau Casio FX-83 neu FX-85 YN UNIG a ganiateir.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

Time allowed - 2 hours

- All questions may be attempted.
- Performance in Section B will be given greater consideration in assigning a first class mark.
- Casio FX-83 or FX-85 calculators ONLY may be used.
- Students may submit answers to this paper in either Welsh or English.

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Section A

1. A model for the growth of a population is

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN\left(1 - \frac{N}{K}\right)$$

where r and K are positive constants.

(a) Explain what the parameters r and K might represent in a biological system. [2 marks]

(b) Find all equilibria and examine their stability. [5 marks]

2. Consider the discrete map

$$N_{t+1} = g(N_t)$$

for some function g and a population N_t.

(a) Give the definition of an $m-$periodic steady state N^* of g. [2 marks]

(b) Find both steady states of this map if g is the function

$$g(N) = rN(3 - N),$$

where $r > 0$. [2 marks]

(c) Find the value of r for which there is a bifurcation to period 2 solutions. [3 marks]

3. For the following system of differential equations

$$\begin{aligned} \dot{x} &= x - 2y, \\ \dot{y} &= 3x - 4y. \end{aligned}$$

(a) Find the steady state and determine its stability. [4 marks]

(b) Find the general solution of this system. [6 marks]

(c) Sketch the phase plane diagram. [4 marks]

4. For the following system of differential equations

$$\begin{aligned} \dot{x} &= 4x - 2y, \\ \dot{y} &= 6x - y. \end{aligned}$$

(a) Find the steady state and determine its stability. [4 marks]

(b) Determine the nullclines and analyse the behaviour of the trajectories on the nullclines and the coordinate axes. [5 marks]

(c) Hence sketch the phase plane diagram. [3 marks]

5. (a) Describe briefly a biological system that this reaction-diffusion scheme might be modelling if $r > 0, 0 < c_1 < 1, 0 < c_2 < 1$: [2 marks]

$$\frac{\partial u}{\partial t} = u(1 - u + c_1 v) + D_1 \frac{\partial^2 u}{\partial x^2},$$
$$\frac{\partial v}{\partial t} = rv(1 + c_2 u - v) + D_2 \frac{\partial^2 v}{\partial x^2}.$$

(b) In the spatially homogeneous case ($D_1 = D_2 = 0$) the equilibrium points of the scheme are

$$(u^*, v^*) = (0,0), (1,0), (0,1), \left(\frac{1 + c_1}{1 - c_1 c_2}, \frac{1 + c_2}{1 - c_1 c_2}\right).$$

By examining the existence and stability of the equilibria, determine the behaviour of the system for initial conditions $u > 0, v > 0$. [10 marks]

6. Consider the dimensionless reaction-diffusion equation

$$u_t = u(1 - u) + u_{xx}.$$

(a) Obtain the ordinary differential equation for the travelling wave solution $u(x,t) = U(z)$, $z = x - ct$, where c is the wave speed, and express it as a pair of first-order equations. [4 marks]

(b) Find the equilibria and analyse their stability. [4 marks]

(c) Sketch the phase portrait and use it to indicate that there is a travelling wave solution satisfying

$$\lim_{z \to \infty} U(z) = 0 \text{ and } \lim_{z \to -\infty} U(z) = 1.$$

[4 marks]

7. Consider the following chemical reaction scheme:

$$A \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} X, \qquad B \xrightarrow{k_2} Y, \qquad 2X + Y \xrightarrow{k_3} 3X.$$

The concentrations a and b, of chemicals A and B respectively, are kept constant. State the Law of Mass Action and use it to derive a pair of differential equations for the concentrations of X and Y. [6 marks]

Section B

8. This is a model of harvesting of a species u in the presence of a second species v:

$$\dot{u} = u(1-v) - \varepsilon u^2 - \delta,$$
$$\dot{v} = v(u-1),$$

with constants $\delta > 0$ and $0 < \varepsilon < \frac{1}{2}$.

(a) Show that the coexistence steady state of this system is at $(u^*, v^*) = (1, 1-\varepsilon-\delta)$, and find the two single-species steady states, giving the condition on δ and ε necessary for them both to exist. [5 marks]

(b) Show that only the coexistence steady state can be stable, and that this requires $\delta < \varepsilon$. [7 marks]

(c) Draw the phase plane diagram in the case $\delta < \varepsilon$. [6 marks]

9. Consider the discrete map

$$N_{t+1} = f(N_t)$$

for some function f.

(a) Show that the eigenvalue associated with an orbit of period 3 denoted $N_1, N_2, N_3, N_1, N_2, N_3, N_1, \ldots$ is given by

$$\lambda = \prod_{i=1}^{3} f'(N_i).$$

[8 marks]

(b) In the case $f(N) = aN^2/(b^2+N^2)$, where a is a positive constant:

(i) Sketch cobweb diagrams for the cases of small a and large a. [8 marks]

(ii) In the case $a^2 > 4b^2$, give the critical population size below which the population will be driven to extinction. [4 marks]

10. Consider the system

$$\frac{\partial n}{\partial t} = bn\left(1 - \frac{n}{n_0}\right),$$
$$\frac{\partial a}{\partial t} = hn - da,$$

for chemicals n and a and positive constants b, n_0, h and d.

(a) Show that in dimensionless form this system can be written

$$\dot{u} = u(1-u),$$
$$\dot{v} = \gamma(u-v),$$

giving γ in terms of the above constants. [6 marks]

(b) Find all steady states of the system and determine their stability. [6 marks]