PRIFYSGOL ABERYSTWYTH UNIVERSITY

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 2 / SEMESTER 2 EXAMINATIONS

MAI – MEHEFIN / MAY – JUNE 2026

MA37410 – Probability and Stochastic Processes

The questions on this paper are written in English.

Amser a ganiateir - 2 awr

- Gellir rhoi cynnig ar bob cwestiwn.
- Rhoddir mwy o ystyriaeth i berfformiad yn Rhan B wrth bennu marc dosbarth cyntaf.
- Cyfrifianellau Casio FX-83 neu FX-85 YN UNIG a ganiateir.
- Darperir tablau ystadegol.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

Time allowed - 2 hours

- All questions may be attempted.
- Performance in Section B will be given greater consideration in assigning a first class mark.
- Casio FX-83 or FX-85 calculators ONLY may be used.
- Statistical tables will be provided.
- Students may submit answers to this paper in either Welsh or English.

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Section A

1. Suppose that X and Y have joint probability density function (pdf)

$$f(x,y) = 3x \text{ if } 0 < y < x < 1 \text{ (and } 0 \text{ elsewhere).}$$

(a) Sketch the region where $f(x,y)$ is positive and find the marginal densities of X and Y. [5 marks]

(b) Calculate the expectations $\mathbb{E}X$ and $\mathbb{E}Y$. [6 marks]

(c) Find the conditional density function $f_{Y|X=x}(y)$ and the conditional expectation $\mathbb{E}(Y|X=x)$. [6 marks]

(d) Verify that $\mathbb{E}\,\mathbb{E}[Y|X] = \mathbb{E}Y$. [3 marks]

2. We throw a biased coin, with probability 0.7 for head, independently N times, where N is a random variable independent of the coin flips with expectation 100 and standard deviation 40. For the total number T of heads find

(a) the expectation $\mathbb{E}T$ [3 marks]

(b) and the standard deviation σ_T. [7 marks]

3. (a) Let X be a random variable that takes values in the non-negative integers. State the definition of the probability generating function (pgf) $G_X(s)$ for X. [2 marks]

(b) Show that $\mathbb{E}X = G'_X(1)$ (evaluation of the derivative at 1). [3 marks]

(c) Compute from the definition the pgf of a Poisson distribution with rate λ (that is: for $n \in \mathbb{N}_0$ the probability of n is $e^{-\lambda}\frac{\lambda^n}{n!}$). [3 marks]

(d) Use (b) and (c) to show that the expectation of a Poisson distribution with rate λ is equal to λ. [2 marks]

4. (a) State the Central Limit Theorem for an i.i.d.-sequence $X_1, X_2, \ldots$ of random variables. What does the abbreviation i.i.d. mean? [4 marks]

(b) We throw a biased coin, with probability 0.7 for head, independently 10000 times. Make use of the Central Limit Theorem to find an approximation of the probability that the number of heads is between 6950 and 7000. Use statistical tables to get an approximate numerical value. [6 marks]

5. Consider a Galton-Watson process with a single ancestor in generation zero and with a family size distribution p_k (for k the number of offspring of an individual) given by $p_0 = p_1 = p_2 = 0.3$, $p_3 = 0.1$.

(a) What is the expected number of offspring per individual? What do you conclude from that about the probability of ultimate extinction? [3 marks]

(b) What is the probability that the process is extinct after one step? Living after one step but extinct after two steps? [4 marks]

(c) Calculate the probability that the process is ultimately extinct. State the result about Galton-Watson processes which you use for your calculation. [5 marks]

(d) How does the probability of ultimate extinction change if we have two ancestors in generation zero instead of one? [2 marks]

6. Consider a Markov chain $X_0, X_1, X_2, \ldots$ with the transition matrix (for the states A, B, C, top to bottom, left to right)

$$P = \begin{pmatrix} 0.5 & 0.2 & 0.3 \\ 0.3 & 0.4 & 0.3 \\ 0.4 & 0.1 & 0.5 \end{pmatrix}.$$

What is the probability that

(a) if $X_0 = A$ then $X_1 = B$?

(b) if $X_0 = A$ then $X_1 = B$ and then $X_2 = C$?

(c) if $X_0 = A$ then $X_2 = C$?

(d) given we start in A (that is $X_0 = A$) we have $X_1 = X_2 = X_3 = X_4 = X_5 = A$?

[6 marks]

Section B

7. Suppose that we have a Galton-Watson process $(X_n)_{n\geq 0}$ with a single ancestor in generation zero and with a family size distribution associated to a random variable Z.

By conditioning X_{n+1} with respect to X_n and making use of standard results about conditional expectation and conditional variance deduce the following recursive formulas:

$$\begin{aligned}\mathbb{E}X_{n+1} &= \mathbb{E}Z \times \mathbb{E}X_n \\ \mathrm{Var}(X_{n+1}) &= (\mathbb{E}Z)^2 \times \mathrm{Var}(X_n) + \mathrm{Var}(Z) \times \mathbb{E}X_n\end{aligned}$$ [6 marks]

8. (a) State the definition of convergence in probability. [2 marks]

(b) Suppose a sequence (X_n) of random variables can be written $X_n = X + Z_n$ so that both the expectations and variances of Z_n converge to 0 for $n \to \infty$. Prove from the definition that (X_n) converges to X in probability, for $n \to \infty$. [8 marks]

9. A Markov chain has the following transition matrix $P = (p_{ij})_{i,j=1,\ldots,6}$ (with $1, \ldots, 6$ top to bottom and left to right).

$$\begin{bmatrix} 0 & 0.2 & 0.8 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0.6 & 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0.7 & 0.3 & 0 \end{bmatrix}$$

(a) Find the communicating classes of states. Which states are recurrent or transient? [5 marks]

(b) Prove that $\lim_{n\to\infty} p_{16}^{(n)}$ (the long-time transition probability from state 1 to state 6) exists, by stating a condition which ensures convergence and then explicitly verifying the condition in this example. [5 marks]

(c) Compute $\lim_{n\to\infty} p_{16}^{(n)}$. [10 marks]

(d) If we start in state 4, what is the expectation for the number of steps needed to return to state 4 for the first time? [7 marks]

10. Suppose that the transition matrix of a finite Markov chain with states $1, \ldots, d$ is symmetric, that is $p_{ij} = p_{ji}$ for all states i, j. Prove that in this case the uniform distribution ($\pi_j = 1/d$ for all j) is stationary. [7 marks]