PRIFYSGOL ABERYSTWYTH UNIVERSITY

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 2/ SEMESTER 2 EXAMINATIONS

MAI / MAY 2026

MA38010 - Spectral Theory

The questions on this paper are written in English.

Amser a ganiateir - 2 awr

- Gellir rhoi cynnig ar bob cwestiwn.
- Rhoddir mwy o ystyriaeth i berfformiad yn rhan B wrth bennu marc dosbarth cyntaf.
- Cyfrifianellau Casio FX-83 neu FX-85 YN UNIG a ganiateir.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

Time allowed - 2 hours

- All questions may be attempted.
- Performance in section B will be given greater consideration in assigning a first class mark.
- Casio FX-83 or FX-85 calculators ONLY may be used.
- Students may submit answers to this paper in either Welsh or English.

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Section A

1. (a) Let $(X, \langle ., . \rangle)$ be an inner product space.

(i) Explain what is meant by saying that x and y are *orthogonal*, where $x, y \in X$. [2 marks]

(ii) Let $\|.\|$ denote the induced norm on X. Suppose that $x, y \in X$ are orthogonal vectors with $\|x\| = 4$, $\|x + y\| = 5$; calculate $\|y\|$. [4 marks]

(b) Let $\omega = e^{2\pi i/3}$.

(i) Demonstrate that $(1, \omega, \omega^2)$ and $(1, \omega^2, \omega)$ are orthogonal vectors in $\mathbb{C}^3$, with respect to the standard inner product on $\mathbb{C}^3$. [3 marks]

(ii) Find the vector in $\text{sp}((1, \omega, \omega^2), (1, \omega^2, \omega))$ that is nearest to $(1, 1, 0)$. [6 marks]

2. Let W be the subspace of $\mathbb{R}^4$ defined by

$$W = \text{sp}((1, 0, 0, 1), (3, 1, 1, 0), (0, 2, 1, 0)).$$

Use Gram-Schmidt orthogonalisation to find an orthogonal basis $\{u_1, u_2, u_3\}$ for W, with respect to the standard inner product on $\mathbb{R}^4$. [6 marks]

3. Suppose that $M = \text{sp}((1, 0, 2, 0), (0, 1, 0, -2)) \subset \mathbb{R}^4$. Calculate $M^{\perp}$, the orthogonal complement of M (with respect to the standard inner product on $\mathbb{R}^4$). [4 marks]

4. Let $l^2_{\mathbb{C}}$ denote the vector space of square summable complex sequences. Define $B : l^2_{\mathbb{C}} \to l^2_{\mathbb{C}}$ by

$$B(x) = (3x(2), x(3), x(4), ..., x(n), ...),$$

where $x = (x(1), x(2), x(3), ...) \in l^2_{\mathbb{C}}$.

(a) Demonstrate that B is linear and bounded; calculate $\|B\|$, the operator norm of B. [2,3,2 marks]

(b) Show that B is surjective, but not injective. [2,2 marks]

(c) Calculate B^*, the adjoint of B. [3 marks]

5. Let $l^2_{\mathbb{R}}$ denote the vector space of square summable real sequences. Define $T : l^2_{\mathbb{R}} \to l^2_{\mathbb{R}}$ by

$$T(x) = (x(1)/4, x(2)/5, ..., x(n)/(n+3), ...),$$

where $x = (x(1), x(2), x(3), ...) \in l^2_{\mathbb{R}}$.

(a) Show that T is linear and bounded; calculate $\|T\|$, the operator norm of T. [2,3,2 marks]

(b) Demonstrate that T is self-adjoint. [2 marks]

(c) Prove that T is compact. [7 marks]

(You may state any results you require about compact operators without proof.)

(d) Calculate the eigenvalues of T. [4 marks]

(e) Verify T is equal to the representation given in the spectral theorem for compact self-adjoint operators. [2 marks]

6. (a) Let H be a Hilbert space, and suppose that $A \in \mathcal{B}(H)$, the space of bounded linear mappings from H to itself. What is meant by the *spectrum* of A, denoted $\sigma(A)$, and the *point spectrum* of A, denoted $\sigma_p(A)$? Demonstrate that $\sigma_p(A) \subset \sigma(A)$. [4 marks]

(b) Let $l^2_{\mathbb{C}}$ denote the vector space of square summable complex sequences. Let $A \in \mathcal{B}(l^2_{\mathbb{C}})$ be defined by

$$A(x) = (x(1)/\sqrt{6}, x(2)/\sqrt{7}, ..., x(n)/\sqrt{n+5}, ...).$$

Demonstrate that $0 \in \sigma(A) \backslash \sigma_p(A)$. [5 marks]

SECTION B BEGINS ON THE NEXT PAGE

Section B

In this section you may use without proof the result that for a closed subspace M of a Hilbert space H, $M \oplus M^{\perp} = H$.

7. Let $(X, \langle .,. \rangle)$ be an inner product space. Prove *Bessel's Inequality*: if $(u_n)_{n=1}^{\infty}$ is an orthonormal sequence in X, show that

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|^2 \text{ for every } x \in X,$$

where $\|.\|$ denotes the induced norm. [7 marks]

8. Let $(X, \langle .,. \rangle)$ be an inner product space. Suppose that $S \in \mathcal{B}(X)$ satisfies $\langle S(x), x \rangle \geq 0$ for every $x \in X$, where $\mathcal{B}(X)$ denotes the vector space of bounded linear mappings from X to itself. Demonstrate Ker$(I + S) = \{0\}$, where I denotes the identity on X, and Ker$(I + S)$ denotes the kernel of $I + S$. [4 marks]

9. Prove the *Riesz Representation Theorem*: Let H be a Hilbert space:

(a) For $z \in H$, define $T(x) = \langle x, z \rangle$ for each $x \in H$. Show that $T \in H^*$, where H^* denotes the dual space of H, and that $\|T\| = \|z\|$. [4 marks]

(b) Prove that for any $z^* \in H^*$ there exists exactly one $z \in H$ such that $z^*(x) = \langle x, z \rangle$ for each $x \in H$. [8 marks]

10. Let φ be a non-zero vector in a Hilbert space H. Define $T : H \to H$ by $T(x) = \langle 2x, \varphi \rangle \varphi$ for $x \in H$.

(a) Show that $T \in \mathcal{B}(H)$, and calculate $\|T\|$. [4,2 marks]

(b) Demonstrate that T is self-adjoint. [3 marks]

11. (a) Let X be a Banach space, and suppose $T \in \mathcal{B}(X)$ satisfies $\|T\| = 1/3$. Write $S = \sum_{n=0}^{\infty} T^n$, where T^0 is the identity mapping I. Prove that $S \circ (I - T) = I$, where $\circ$ denotes composition of functions. [9 marks]

(You may use the result that an absolutely convergent series in a Banach space is convergent without proof.)

(b) For $A, B \in \mathcal{B}(l^2_{\mathbb{C}})$, does $A \circ B = I$ imply that A and B are invertible, where I denotes the identity mapping? Give a proof or counterexample as appropriate to your answer. [3 marks]

12. Let H be an infinite dimensional Hilbert space, and suppose $A \in \mathcal{B}(H)$ satisfies $A^3 = I$, where I denotes the identity mapping (on H). Prove that A is not compact. [6 marks]

END OF EXAM PAPER