PRIFYSGOL ABERYSTWYTH UNIVERSITY

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 2 / SEMESTER 2 EXAMINATIONS

MAI – MEHEFIN / MAY – JUNE 2026

MA38310 – Operator Algebra

The questions on this paper are written in English.

Amser a ganiateir - 2 awr

- Gellir rhoi cynnig ar bob cwestiwn.
- Rhoddir mwy o ystyriaeth i berfformiad yn Rhan B wrth bennu marc dosbarth cyntaf.
- Cyfrifianellau Casio FX-83 neu FX-85 YN UNIG a ganiateir.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

Time allowed - 2 hours

- All questions may be attempted.
- Performance in Section B will be given greater consideration in assigning a first class mark.
- Casio FX-83 or FX-85 calculators ONLY may be used.
- Students may submit answers to this paper in either Welsh or English.

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Section A

1. Consider the quaternions $q_1 = 3 + 2i + j - 2k$ and $q_2 = 1 + 4i + 2j + 5k$. Write the following in the form $w + xi + yj + zk$ for some $w, x, y, z \in \mathbb{R}$:

(a) $5q_1 - 2q_2$; (b) $q_1 q_2$; (c) $|iq_2|$; (d) q_1^{-1}.

[10 marks]

2. (a) Let ω be a non-negative real number. Prove that the solution set of the equation $x^2 + \omega^2 = 0$ in $\mathbb{H}$ is $\{x \in \operatorname{Im}\mathbb{H} : |x| = \omega\}$. [6 marks]

(b) Hence, or otherwise, find the solution set of $4x^2 - 8x + 13 = 0$ in $\mathbb{H}$ and state 7 explicit distinct solutions in the form $\alpha + u$ with $\alpha \in \mathbb{R}$ and $u \in \operatorname{Im}\mathbb{H}$. [4 marks]

3. Determine which of the following subsets of the algebra $\mathbb{R}^2$ are subalgebras. Justify your answers clearly.

(a) $S_1 = \{(\alpha + \beta, \alpha - \beta) : \alpha, \beta \in \mathbb{R}\}$;

(b) $S_2 = \{(\alpha, -\alpha) : \alpha \in \mathbb{R}\}$.

[9 marks]

4. Determine which of the following mappings are algebra homomorphisms. For each algebra homomorphism: determine its kernel; is it injective; and is it unital? Justify your answers clearly.

(a) $\pi_1 : \mathbb{R} \to \mathbb{H}, \quad \pi_1(x) = -xi + xj - xk$ for all $x \in \mathbb{R}$;

(b) $\pi_2 : \mathbb{Q} \to M_2(\mathbb{Q}), \quad \pi_2(x) = \frac{1}{5}\begin{pmatrix} 4x & 2x \\ 2x & x \end{pmatrix}$ for all $x \in \mathbb{Q}$.

[12 marks]

5. Recall that the left regular representation of an algebra A is the homomorphism $\rho : A \to \mathcal{L}(A)$, from A into the algebra of linear transformations on A, given by $\rho(a)(b) = ab$ for all $a, b \in A$.

Let $A = \mathbb{C}C_4$, the group algebra of the cyclic group of order 4, $C_4 = \{\mathbb{1}, \omega, \omega^2, \omega^3\}$, over $\mathbb{C}$.

(a) Prove that ρ is injective. [3 marks]

(b) Find the matrix representation of $\rho(a)$, for $a = a_0\mathbb{1} + a_1\omega + a_2\omega^2 + a_3\omega^3 \in A$ ($a_i \in \mathbb{C}$ for $i \in \{0, 1, 2, 3\}$), with respect to the basis C_4. [5 marks]

(c) Hence, or otherwise, give an explicit description of a subalgebra of $M_4(\mathbb{C})$ that is isomorphic to $\mathbb{C}C_4$. [2 marks]

6. Let $v_1 = (i, -2, 0)$, $v_2 = (-i, 1, i)$ and let $M = \operatorname{span}\{v_1, v_2\}$ in the Hilbert space $\mathbb{C}^3$.

(a) Use the Gram-Schmidt process to find an orthonormal basis for M. [6 marks]

(b) Find the matrix representation of the orthogonal projection of $\mathbb{C}^3$ onto M with respect to the standard basis of $\mathbb{C}^3$. [5 marks]

7. Compute the C^*-norms of the following matrices in $M_2(\mathbb{C})$:

(a) $\begin{pmatrix} 2 & -i \\ i & -1 \end{pmatrix}$; (b) $\begin{pmatrix} 2 & 2i \\ i & -1 \end{pmatrix}$.

[8 marks]

Section B

8. (a) Explain what it means for an algebra A to be simple. [3 marks]

(b) Prove that the two-dimensional algebra of complex numbers, $\mathbb{C}$, over the field $\mathbb{R}$, is simple. [5 marks]

9. Consider the subalgebra $A = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$ of $M_2(\mathbb{R})$ as an algebra in its own right (you do not need to prove it is an algebra).

(a) Show that $I = \left\{ \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} : \alpha \in \mathbb{R} \right\}$ is an ideal of A. [4 marks]

(b) Prove that A/I is isomorphic to $\mathbb{R}^2$. [9 marks]

10. Recall that if a is a self-adjoint element in a unital finite-dimensional C^*-algebra A then there is a $*$-isomorphism $\Phi : C(\sigma(a)) \to C^*(a, \mathbb{1})$ satisfying $\Phi(\chi_\lambda) = e_\lambda$ for all $\lambda \in \sigma(a)$ and $\Phi(\mathrm{id}) = a$, where $\sigma(a)$ is the spectrum of a, χ_λ is the characteristic function of λ and e_λ is the spectral projection of a associated with λ.

Consider the self-adjoint matrix

$$a = \begin{pmatrix} 7 & 4i \\ -4i & 1 \end{pmatrix}$$

in $M_2(\mathbb{C})$, the C^*-algebra of 2×2 matrices with complex entries.

Determine the matrix entries of $\Phi(f)$ where $f : \sigma(a) \to \mathbb{C}$ is defined by $f(x) = \sqrt{|x|}$. [9 marks]

11. Recall von Neumann's Density Theorem in finite dimensions:

Let A be a unital involutive subalgebra of $M_n(\mathbb{C})$. Then $A = A''$, i.e. A is a von Neumann algebra on $\mathbb{C}^n$.

Prove von Neumann's Density Theorem in finite dimensions by proving each of the following steps, or otherwise.

Let A be an involutive subalgebra of $M_n(\mathbb{C})$, which contains the identity matrix. Let K be the direct sum of n copies of $\mathbb{C}^n$ and let $\pi : M_n(\mathbb{C}) \to \mathcal{B}(K)$ be the representation of $M_n(\mathbb{C})$ on K given by the diagonal action. Identify $\mathcal{B}(K)$ with $M_n(M_n(\mathbb{C}))$ in the usual way.

(a) Prove that $\pi(A)' = M_n(A')$.

(b) Prove that $\pi(A'') \subseteq M_n(A')' = \pi(A)''$.

(c) Suppose that S is a non-empty subset of $\mathcal{B}(K)$, M is a subspace of K and p is the orthogonal projection of K onto M. Prove that

(i) if S is self-adjoint and M is S-invariant then $p \in S'$; and

(ii) if $p \in S'$ then M is S-invariant.

(d) Prove that $A'' = A$.
[*Hint: Consider the subspace $\pi(A)(v_1 \oplus v_2 \oplus \cdots \oplus v_n)$ of K, where $\{v_i\}_{i=1}^n$ is an orthonormal basis of $\mathbb{C}^n$.*]

[20 marks]