PRIFYSGOL ABERYSTWYTH UNIVERSITY

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 2/ SEMESTER 2 EXAMINATIONS

MAI / MAY 2026

MAM8020 - Spectral Theory

The questions on this paper are written in English.

Amser a ganiateir - 2 awr

- Gellir rhoi cynnig ar bob cwestiwn.
- Rhoddir mwy o ystyriaeth i berfformiad yn rhan B wrth bennu marc dosbarth cyntaf.
- Cyfrifianellau Casio FX-83 neu FX-85 YN UNIG a ganiateir.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

Time allowed - 2 hours

- All questions may be attempted.
- Performance in section B will be given greater consideration in assigning a first class mark.
- Casio FX-83 or FX-85 calculators ONLY may be used.
- Students may submit answers to this paper in either Welsh or English.

© 2026 Aberystwyth University

Section A

1. (a) Let $(X, \langle ., . \rangle)$ be an inner product space. Explain what is meant by saying that x and y are *orthogonal*, where $x, y \in X$. [2 marks]

(b) Let $\omega = e^{2\pi i/3}$.

(i) Demonstrate that $(1, \omega, \omega^2)$ and $(1, \omega^2, \omega)$ are orthogonal vectors in $\mathbb{C}^3$, with respect to the standard inner product on $\mathbb{C}^3$. [3 marks]

(ii) Find the vector in $\text{sp}((1, \omega, \omega^2), (1, \omega^2, \omega))$ that is nearest to $(1, 1, 0)$. [6 marks]

2. Let W be the subspace of $\mathbb{R}^4$ defined by

$$W = \text{sp}((1, 0, 0, 1), (3, 1, 1, 0), (0, 2, 1, 0)).$$

Use Gram-Schmidt orthogonalisation to find an orthogonal basis $\{u_1, u_2, u_3\}$ for W, with respect to the standard inner product on $\mathbb{R}^4$. [6 marks]

3. Suppose that $M = \text{sp}((1, 0, 2, 0), (0, 1, 0, -2)) \subset \mathbb{R}^4$. Calculate $M^{\perp}$, the orthogonal complement of M (with respect to the standard inner product on $\mathbb{R}^4$). [4 marks]

4. Let $l^2_{\mathbb{R}}$ denote the vector space of square summable real sequences. Define $T : l^2_{\mathbb{R}} \to l^2_{\mathbb{R}}$ by

$$T(x) = (x(1)/4, x(2)/5, ..., x(n)/(n + 3), ...),$$

where $x = (x(1), x(2), x(3), ...) \in l^2_{\mathbb{R}}$.

(a) Show that T is linear and bounded; calculate $\|T\|$, the operator norm of T. [2,3,2 marks]

(b) Demonstrate that T is self-adjoint. [2 marks]

(c) Prove that T is a compact operator. [7 marks]

(You may state any results you require about compact operators without proof.)

(d) Calculate the eigenvalues of T. [4 marks]

(e) Verify T is equal to the representation given in the spectral theorem for compact self-adjoint operators. [2 marks]

5. (a) Let H_1 and H_2 be Hilbert spaces. Explain what is meant by saying that $A \in \mathcal{B}(H_1, H_2)$ is a *Hilbert-Schmidt operator*, and define $\|.\|_{HS}$, the *Hilbert-Schmidt norm*. Here $\mathcal{B}(H_1, H_2)$ denotes the space of bounded linear mappings from H_1 to H_2. [3 marks]

(b) Define $A \in \mathcal{B}(l^2_{\mathbb{R}})$, the space of bounded linear mappings from $l^2_{\mathbb{R}}$ to itself, by

$$A(x) = (0, 0, x(3)/(\sqrt{3}(\ln 3)^2), x(4)/(2(\ln 4)^2), ..., x(n)/(\sqrt{n}(\ln n)^2), ...).$$

Demonstrate that A is a Hilbert-Schmidt operator. [7 marks]

6. (a) Let H be a Hilbert space, and suppose that $A \in \mathcal{B}(H)$, the space of bounded linear mappings from H to itself. What is meant by the *spectrum* of A, denoted $\sigma(A)$, and the *point spectrum* of A, denoted $\sigma_p(A)$? Demonstrate that $\sigma_p(A) \subset \sigma(A)$. [4 marks]

(b) Let $l^2_{\mathbb{C}}$ denote the vector space of square summable complex sequences. Let $A \in \mathcal{B}(l^2_{\mathbb{C}})$ be defined by

$$A(x) = (x(1)/\sqrt{6}, x(2)/\sqrt{7}, ..., x(n)/\sqrt{n+5}, ...).$$

(i) Demonstrate that $0 \in \sigma(A) \backslash \sigma_p(A)$. [5 marks]

(ii) Show that A is not a Hilbert-Schmidt operator. [3 marks]

7. (a) State the form of the differential operator for a *regular Sturm-Liouville* (RSL) problem on $[0, 1]$. [2 marks]

(b) For $y = y(t)$, and writing y' for the derivative of y with respect to t, show that the differential operator $\mathcal{L}(y) = y'' + 3y'$ does not yield a RSL problem; find a function $\alpha = \alpha(t)$ such that $\alpha\mathcal{L}$ does give a RSL problem. [3 marks]

SECTION B BEGINS ON THE NEXT PAGE

Section B

In this section you may use without proof the result that for a closed subspace M of a Hilbert space H, $M \oplus M^{\perp} = H$.

8. Let $(X, \langle ., . \rangle)$ be an inner product space. Prove *Bessel's Inequality*: if $(u_n)_{n=1}^{\infty}$ is an orthonormal sequence in X, show that

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|^2 \text{ for every } x \in X,$$

where $\|.\|$ denotes the induced norm. [7 marks]

9. Let $(X, \langle ., . \rangle)$ be an inner product space. Suppose that $S \in \mathcal{B}(X)$ satisfies $\langle S(x), x \rangle \geq 0$ for every $x \in X$, where $\mathcal{B}(X)$ denotes the vector space of bounded linear mappings from X to itself. Demonstrate Ker$(I + S) = \{0\}$, where I denotes the identity on X, and Ker$(I + S)$ denotes the kernel of $I + S$. [4 marks]

10. Prove the *Riesz Representation Theorem*: Let H be a Hilbert space:

(a) For $z \in H$, define $T(x) = \langle x, z \rangle$ for each $x \in H$. Show that $T \in H^*$, where H^* denotes the dual space of H, and that $\|T\| = \|z\|$. [3 marks]

(b) Prove that for any $z^* \in H^*$ there exists exactly one $z \in H$ such that $z^*(x) = \langle x, z \rangle$ for each $x \in H$. [8 marks]

11. Let φ be a non-zero vector in a Hilbert space H. Define $T \in \mathcal{B}(H)$ by $T(x) = \langle 2x, \varphi \rangle \varphi$ for $x \in H$. Demonstrate that T is self-adjoint. [3 marks]

12. Let H be an infinite dimensional Hilbert space, and suppose $A \in \mathcal{B}(H)$ satisfies $A^3 = I$, where I denotes the identity mapping (on H). Prove that A is not compact. [6 marks]

13. Define an integral operator $S \in \mathcal{B}(L^2(0,1))$, where $L^2(0,1)$ denotes the space of square integrable functions on $(0,1)$, by

$$S(f)(t) = \int_0^t s^2 f(s) ds \text{ for } f \in L^2(0,1),\ 0 < t < 1.$$

Prove that S is a Hilbert-Schmidt operator, and calculate the Hilbert-Schmidt norm $\|S\|_{HS}$. [9,1 marks]

14. Let $\mathcal{L}$ be the differential operator for a regular Sturm-Liouville (RSL) system. An *eigenvalue* λ satisfies $\mathcal{L}(f) = -\lambda \rho f$ for a non-zero *eigenfunction* f (where continuous $\rho > 0$); eigenvalues are real numbers, and eigenfunctions corresponding to distinct eigenvalues are orthogonal with respect to the weighted L^2 inner product

$$\langle f, g \rangle = \int_a^b \rho(t) f(t) \overline{g(t)} dt.$$

Prove that the set of eigenvalues of a RSL is countable. [9 marks]

END OF EXAM PAPER