PRIFYSGOL ABERYSTWYTH UNIVERSITY

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 2 / SEMESTER 2 EXAMINATIONS

MAI – MEHEFIN / MAY – JUNE 2026

MP11010 – Further Algebra and Calculus

The questions on this paper are written in English.

Amser a ganiateir - 2 awr

- Rhoddir marciau llawn am atebion cyflawn i bob cwestiwn yn Rhan A ac i dri cwestiwn yn Rhan B.
- Yn Rhan B, rhoddir credyd am y tri ateb gorau.
- Cyfrifianellau Casio FX-83 neu FX-85 YN UNIG a ganiateir.
- Mae taflen fformiwlâu wedi cael ei hatodi.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

Time allowed - 2 hours

- Full marks will be given for complete answers to all questions in Section A and to three questions in Section B.
- In Section B, credit will be given for the best three answers.
- Casio FX-83 or FX-85 calculators ONLY may be used.
- A formula sheet is attached.
- Students may submit answers to this paper in either Welsh or English.

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Section A

1. Let

$$\underline{\mathbf{A}} = \begin{bmatrix} -7 \\ 11 \end{bmatrix}, \quad \underline{\mathbf{B}} = \begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}, \quad \underline{\mathbf{C}} = \begin{bmatrix} -1 & 1 \\ 0 & 2 \\ 3 & 0 \end{bmatrix}.$$

Determine which (if any) of the following matrices are well defined:

(i) $\underline{\mathbf{A}}\,\underline{\mathbf{B}}\,\underline{\mathbf{C}}$; (ii) $\underline{\mathbf{B}}^{-1}\,\underline{\mathbf{C}}^{\mathsf{T}}$; (iii) $\underline{\mathbf{C}}\,\underline{\mathbf{B}}^{\mathsf{T}} + \underline{\mathbf{B}}\,\underline{\mathbf{C}}^{\mathsf{T}}$.

Give an explanation where they are not well defined, and compute those that are. [5 marks]

2. Let

$$\underline{\mathbf{A}} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}.$$

Find the inverse of $\underline{\mathbf{A}}$ using Gauss-Jordan Elimination. [5 marks]

3. Use Cramer's rule to solve the following linear system of equations:

$$\begin{cases} 3x - y &= 9; \\ 5x + 2y &= 4. \end{cases}$$

[5 marks]

4. Let

$$\underline{\mathbf{A}} = \begin{bmatrix} 3 & 0 \\ -4 & -2 \end{bmatrix}.$$

Identify the eigenvalues of $\underline{\mathbf{A}}$ and find corresponding eigenvectors. [5 marks]

5. Let $f(x, y) = \dfrac{1}{2xy} - x \sin 3y$, for $x, y \neq 0$.

Determine the partial derivatives $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial^2 f}{\partial x^2}, \dfrac{\partial^2 f}{\partial y^2}$, and show that $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$.

[5 marks]

6. Consider the following change of variables:

$$\begin{cases} x = \mathrm{e}^u \cos v, \\ y = \mathrm{e}^u \sin v. \end{cases}$$

Find (in terms of u and v) the Jacobian matrix $\underline{\mathbf{J}} = \underline{\mathbf{J}}(x, y; u, v)$ and its determinant (in its simplest form). [5 marks]

7. Find the linear approximation of $f(x, y) = \dfrac{1}{\sqrt{x^2 + y^2}}$ near $(3, -4)$. [5 marks]

8. Evaluate the double integral

$$I = \int_{x=-1}^{2} \int_{y=x^2}^{x+2} 10y \, \mathrm{d}y \, \mathrm{d}x.$$

[5 marks]

Section B

9. Consider the following set of linear equations (1):

$$\begin{cases} x + 2y - z & = 4, \\ 2x - y + 3z & = 7, \\ 3x + ky + 2z & = a. \end{cases} \tag{1}$$

(a) Let $k = 2, a = 11$.

(i) Find the determinant of the coefficient matrix and explain why Cramer's Rule can be used to find the solution to the system of equations (1). [3 marks]

(ii) Find the solution using Cramer's Rule. [6 marks]

(b) (i) Show that for $k = 1$ the equations (1) have no unique solution. [3 marks]

(ii) Find all the solutions to the equations (1) using Gaussian Elimination for $k = 1, a = 11$. [5 marks]

(iii) Show that the equations (1) have no solutions for $k = 1, a = 4$. [3 marks]

10. (a) Let $f(x, y) = 2x^3 + xy^2 - 6x^2 - 3y^2$. Find the critical point(s) of f and determine their nature. [12 marks]

(b) Let

$$\underline{\mathbf{A}} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 0 \\ -1 & 2 & 3 \end{bmatrix}.$$

Find the inverse of $\underline{\mathbf{A}}$ using the Laplace expansion method. [8 marks]

11. (a) Let

$$\underline{\mathbf{A}} = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}.$$

(i) Find the eigenvalues of $\underline{\mathbf{A}}$. [4 marks]

(ii) Verify the eigenvalues using the determinant or trace of $\underline{\mathbf{A}}$. [2 marks]

(iii) Find **unit** eigenvectors of $\underline{\mathbf{A}}$ corresponding to the eigenvalues. [4 marks]

(b) Consider the integral

$$I = \int_{x=0}^{1} \int_{y=0}^{x^2} \frac{1}{\sqrt{y}} \, \mathrm{d}y \, \mathrm{d}x.$$

(i) Explain why the integral is *improper*. [2 marks]

(ii) Evaluate the integral. [8 marks]

12. (a) Let

$$\underline{\mathbf{A}} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}; \quad \underline{\mathbf{B}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}; \quad \underline{\mathbf{C}} = \begin{bmatrix} 2 & -4 \\ 3 & -6 \end{bmatrix};$$

and $\mathrm{S}_{BA}, \mathrm{S}_{CBA}$ be images of the unit square (with its lower left vertex located at the origin) under the transformations $\underline{\mathbf{B}}\,\underline{\mathbf{A}}$ and $\underline{\mathbf{C}}\,\underline{\mathbf{B}}\,\underline{\mathbf{A}}$ respectively.

(i) State the eigenvalues and eigenvectors of $\underline{\mathbf{A}}$. [2 marks]

(ii) Show that $\underline{\mathbf{B}}$ is orthogonal. [2 marks]

(iii) Find the area of S_{BA}. [2 marks]

(iv) Sketch the shape S_{BA} in the xy-plane. [2 marks]

(v) Explain why the area of S_{CBA} is zero. [2 marks]

(b) Let

$$I = \int_{y=0}^{2} \int_{x=y^2}^{4} \frac{y}{x} \mathrm{e}^{-x} \mathrm{d}x \, \mathrm{d}y.$$

Sketch the domain of integration. Then reverse the order of integration, and hence evaluate the integral. [10 marks]

13. A solid S occupies the volume bounded by the surfaces $z = 0, z = 4 - x^2 - y^2$, and the cylinder $x^2 + y^2 = 1$. It has density $\rho(x, y, z) = 6z$.

(a) To find the mass M of the solid S, consider a change of variables:

$$\begin{cases} x = r\cos\theta; \\ y = r\sin\theta; \\ z = z. \end{cases}$$

Then:

(i) express the bounds of the solid in this coordinate system; [2 marks]

(ii) compute the determinant of the Jacobian $\underline{\mathbf{J}}(x, y; r, \theta)$; [2 marks]

(iii) hence show that (with appropriate limits)

$$M = \iiint_V 6z\, r \, \mathrm{d}z \, \mathrm{d}r \, \mathrm{d}\theta;$$

[2 marks]

(iv) determine the limits of the integral; [2 marks]

(v) solve the triple integral to find $M = 37\pi$. [6 marks]

(b) Find the z component of the centre-of-mass of S. [6 marks]

Useful Formulae

Trigonometric identities

$$\sin^2\theta + \cos^2\theta = 1 \iff \tan^2\theta + 1 = \sec^2\theta$$
$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$
$$\iff \sin^2\theta = (1 - \cos 2\theta)/2 \iff \cos^2\theta = (1 + \cos 2\theta)/2$$
$$\sin 2\theta = 2\sin\theta\cos\theta = (\sin\theta + \cos\theta)^2 - 1$$

Standard derivatives and integrals

$$\frac{\mathrm{d}x^n}{\mathrm{d}x} = nx^{n-1} \qquad \int x^n\,\mathrm{d}x = \tfrac{1}{n+1}x^{n+1} + c,\ n \neq -1$$
$$\frac{\mathrm{d}\mathrm{e}^{kx}}{\mathrm{d}x} = k\mathrm{e}^{kx} \qquad \int \mathrm{e}^{kx}\,\mathrm{d}x = \tfrac{1}{k}\mathrm{e}^{kx} + c$$
$$\frac{\mathrm{d}\ln x}{\mathrm{d}x} = \frac{1}{x} \qquad \int \frac{1}{x}\,\mathrm{d}x = \ln|x| + c$$
$$\frac{\mathrm{d}\sin x}{\mathrm{d}x} = \cos x \qquad \int \cos x\,\mathrm{d}x = \sin x + c$$
$$\frac{\mathrm{d}\cos x}{\mathrm{d}x} = -\sin x \qquad \int \sin x\,\mathrm{d}x = -\cos x + c$$
$$\frac{\mathrm{d}\tan x}{\mathrm{d}x} = \sec^2 x \qquad \int \sec^2 x\,\mathrm{d}x = \tan x + c$$
$$\frac{\mathrm{d}\arcsin(x/a)}{\mathrm{d}x} = \frac{1}{\sqrt{a^2 - x^2}} \qquad \int \frac{1}{\sqrt{a^2 - x^2}}\,\mathrm{d}x = \arcsin(x/a) + c$$
$$\frac{\mathrm{d}\arccos(x/a)}{\mathrm{d}x} = \frac{-1}{\sqrt{a^2 - x^2}} \qquad \int \frac{1}{\sqrt{a^2 - x^2}}\,\mathrm{d}x = -\arccos(x/a) + c$$
$$\frac{\mathrm{d}\arctan(x/a)}{\mathrm{d}x} = \frac{a}{x^2 + a^2} \qquad \int \frac{1}{x^2 + a^2}\,\mathrm{d}x = \tfrac{1}{a}\arctan(x/a) + c$$