PRIFYSGOL ABERYSTWYTH UNIVERSITY

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**DEPARTMENT OF PHYSICS**

**SEMESTER 1 EXAMINATIONS, JANUARY 2026**

**PHM3010 – Advanced Quantum Theory**

**Time allowed – 2 hours**

- **Full marks can be attained for complete answers to ALL six questions in Section A and TWO questions from Section B**
- **The numbers in brackets after the questions indicate the allocation of marks for each question.**
- **Only calculators from the Casio FX83, FX85 or FX991 range may be used.**
- **A booklet of mathematical formulae and standard constants will be provided.**
- **Only a 1 side of A4 size Summary Sheet is allowed.**
- **MAE MODD I FYFYRWYR GYFLWYNO ATEBION I'R PAPUR HWN NAILL AI YN Y GYMRAEG NEU'R SAESNEG**
- **STUDENTS MAY SUBMIT ANSWERS TO THIS PAPER IN EITHER WELSH OR ENGLISH**

**Once seated, students may complete the front cover of the answer book(s) and the attendance slip ONLY.**

**DO NOT TURN THE EXAMINATION QUESTION PAPER OVER UNTIL INSTRUCTED TO DO SO.**

## Section A

**Attempt ALL questions in the booklets provided**

**1** The density matrix for a two-level atom is given by [5]

$$\varrho = \begin{bmatrix} \rho_{ee} & \rho_{eg} \\ \rho_{ge} & \rho_{gg} \end{bmatrix}.$$

State the requirements that the entries must meet.

**2** Describe the quantum mechanical phenomenon of scattering off a one-dimensional potential well. Contrast this with classical mechanics. Describe the phenomenon of transparency. [5]

**3** Describe the time-independent version of Perturbation Theory. Which (multiple) power series is it based on? What is time-independent Perturbation Theory capable of approximating? [5]

**4** Describe time-dependent Perturbation Theory. In what aspect is it different from time-independent Perturbation Theory? Describe applying time-dependent Perturbation Theory, using a perturbation oscillating with time, to model the interaction of an atom with an electromagnetic wave. [5]

**5** Write down the potential $V(r)$ of a 3-dimensional Harmonic Oscillator. [5]

Rewrite this in terms of $x, y, z$ (i.e. $V(x, y, z)$) and show this potential is separable, i.e. it can be split as follows:

$$V(x, y, z) = V_x(x) + V_y(y) + V_z(z)$$

State the energy levels $E_{n_x n_y n_z}$ of the 3-dimensional Harmonic Oscillator in terms of the quantum numbers $n_x, n_y, n_z$.

**6** State Fermi's Golden Rule for a system transitioning to a continuum of energy levels, in terms of a density of states $\rho(E)$ and the matrix element $V_{fj}$ of the perturbing operator $\hat{V}$. Describe its significance. Describe how Fermi's Golden Rule changes when considering an atom interacting with a non-monochromatic electromagnetic wave. [5]

## Section B

## Answer TWO questions

Q7 A two-level atom has a ground state and an excited state given respectively by

$$|g\rangle = \begin{bmatrix}0\\1\end{bmatrix}, \qquad |e\rangle = \begin{bmatrix}1\\0\end{bmatrix}.$$

Its Hamiltonian is given by

$$H = \hbar\omega|e\rangle\langle e| = \hbar\omega \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}.$$

a) The master equation for the closed system is

$$\frac{d\varrho}{dt} = \frac{1}{i\hbar}[H, \varrho].$$

Show that the solution is

$$\varrho(t) = \begin{bmatrix}\rho_{ee}(0) & e^{-i\omega t}\rho_{eg}(0)\\ e^{+i\omega t}\rho_{ge}(0) & \rho_{gg}(0)\end{bmatrix}.$$

[2]

Explain briefly what happens to the state as it evolves (you may use the Bloch sphere representation here).

[1]

b) The atom is in contact with a thermal heat bath which should drive the system to the appropriate equilibrium state. Here, the master equation for the open system is the Lindblad form

$$\frac{d\varrho}{dt} = \frac{1}{i\hbar}[H, \varrho] + \sum_k \mathcal{D}_k(\varrho)$$

with the dissipative terms taking the form

$$\mathcal{D}_k(\varrho) = L_k \varrho L_k^* - \frac{1}{2}\varrho L_k^* L_k - \frac{1}{2} L_k^* L_k \varrho$$

and there are two dissipators:

$$L_1 = \sqrt{\gamma(1-\bar{n})}\,\sigma$$

$$L_2 = \sqrt{\gamma\bar{n}}\sigma^*$$

where,

$$\sigma = |g\rangle\langle e| = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Show that

$$\mathcal{D}_1(\varrho) = \gamma(1-\bar{n})\begin{bmatrix} -\rho_{ee} & -\frac{1}{2}\rho_{eg} \\ -\frac{1}{2}\rho_{ge} & \rho_{ee} \end{bmatrix},$$

$$\mathcal{D}_2(\varrho) = \gamma\bar{n}\begin{bmatrix} -\rho_{gg} & -\frac{1}{2}\rho_{eg} \\ -\frac{1}{2}\rho_{ge} & \rho_{gg} \end{bmatrix}.$$

[4]

c) Putting the various terms together, show that the open system density matrix evolves as

$$\frac{d}{dt}\rho_{ee} = -\gamma(\rho_{ee} - \bar{n}),$$

and

$$\frac{d}{dt}\rho_{eg} = -\left(\frac{1}{2}\gamma + i\omega\right)\rho_{eg}.$$

[4]

d) Show that, irrespective of the initial state, the state will always converge to

$$\varrho(\infty) = \begin{bmatrix} \bar{n} & 0 \\ 0 & 1-\bar{n} \end{bmatrix}.$$

[4]

**Q8** Two quantum systems, $A$ and $B$, are governed by a Hamiltonian

$$H = H_A \otimes I_B + I_A \otimes H_B + V.$$

a) State what it means for the initial state of the combined system to be either separable or entangled.

[2]

b) If the initial state is separable and the interaction $V = 0$, show that the two systems never become entangled.

[3]

c) A two-level atom interacts with a cavity mode with the Hamiltonian being given by

$$H = \hbar\Delta\, a^* a + \hbar\omega\, \sigma^* \sigma + \sqrt{\gamma}(a^* \sigma + \sigma^* a).$$

Here, $a$ is the annihilator for the cavity mode, $\sigma$ is lowering operator for the two-level atom, and $\Delta, \omega, \gamma$ are positive constants. (We have dropped the tensor product symbol for convenience.)

Explain what physically is going on in this model.

[2]

Derive the Heisenberg equations of motion for the cavity mode $a$.

[3]

d) If we start off with the cavity in the vacuum state and the two-level atom in its excited state, what should happen after a finite time of interaction. Will the reduced state of the cavity still be pure?

[5]

**Q9** Consider the 1-dimensional harmonic oscillator:

$$\widehat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2$$

The purpose of this question is to estimate the ground-state energy of the harmonic oscillator using the variational principle with the following trial wavefunction:

$$\begin{aligned} \nu(x) &= 0 && \text{if } x < -a \\ \nu(x) &= N(a^2 - x^2) && \text{if } -a \le x \le a \\ \nu(x) &= 0 && \text{if } x > a \end{aligned}$$

where $N$ is a normalisation factor and $a$ is an adjustable parameter.

a) Determine $N$ by normalisation and show that the normalised trial wavefunction is given by:

$$\nu(x) = \sqrt{\frac{15}{16a^5}}(a^2 - x^2).$$

[4]

b) We have:

$$\langle \nu | \widehat{H} | \nu \rangle = -\frac{\hbar^2}{2m}\left\langle \nu \middle| \frac{d^2\nu}{dx^2} \right\rangle + \frac{1}{2}m\omega^2 \langle \nu | x^2 | \nu \rangle$$

Show that:

$$\langle \nu | x^2 | \nu \rangle = \frac{a^2}{7}$$

[3]

c) Show that:

$$\left\langle \nu \left| \frac{d^2\nu}{dx^2} \right\rangle = -\frac{5}{2a^2} \right.$$

[4]

d) Minimize $\langle \nu | \widehat{H} | \nu \rangle$ with respect to $a$ and show that the minimal value is given by:

$$\langle \nu | \widehat{H} | \nu \rangle = \sqrt{\frac{5}{14}} \hbar\omega$$

How does this compare to the exact ground state energy of the 1-dimensional harmonic oscillator?

[4]

**END OF PAPER**