PRIFYSGOL ABERYSTWYTH UNIVERSITY

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DEPARTMENT OF PHYSICS

SEMESTER 1 EXAMINATIONS, JANUARY 2026

PM26020 – Mathematical Physics

Time allowed – 3 hours

- **Full marks can be attained for complete answers to ALL eight questions in Section A and TWO questions from Section B**
- **The numbers in brackets after the questions indicate the allocation of marks for each question.**
- **Only calculators from the Casio FX83, FX85 or FX991 range may be used.**
- **A booklet of mathematical formulae and standard constants will be provided.**
- **Only a 1 side of A4 size Summary Sheet is allowed.**
- **MAE MODD I FYFYRWYR GYFLWYNO ATEBION I'R PAPUR HWN NAILL AI YN Y GYMRAEG NEU'R SAESNEG**
- **STUDENTS MAY SUBMIT ANSWERS TO THIS PAPER IN EITHER WELSH OR ENGLISH**

Once seated, students may complete the front cover of the answer book(s) and the attendance slip ONLY.

DO NOT TURN THE EXAMINATION QUESTION PAPER OVER UNTIL INSTRUCTED TO DO SO.

Section A

Attempt ALL questions in the booklets provided

1 Given the vectors: [5]

$$\boldsymbol{a} = 1\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$$

$$\boldsymbol{b} = 5\boldsymbol{i} + 7\boldsymbol{j} + 9\boldsymbol{k}$$

construct a vector $\boldsymbol{c}$ that is at right angles to $\boldsymbol{a}$ and $\boldsymbol{b}$.

Turn the vector $\boldsymbol{c}$ into a unit vector.

2 Imagine a person using spherical polar coordinates to navigate around the globe, where $\theta = 0$ corresponds to the North pole. Describe the meaning of the coordinates $r,\ \theta,\ \phi$ in terms of latitude, longitude, etc. State the scale factors for the spherical polar coordinate system. Explain the effects that the scale factors may have in geographical terms. [5]

3 Given a vector field: [5]

$$\boldsymbol{a} = xy\boldsymbol{i} + z\cos x\,\boldsymbol{j} + xy^2\boldsymbol{k}$$

calculate $curl\ \boldsymbol{a}$.

Is $\boldsymbol{a}$ a conservative vector field?

4 Evaluate the line integral: [5]

$$\int_C xy\,dx + x^2\,dy$$

along the curve C defined by the equation $y = x^3$ between the points $(0,0)$ and $(1,1)$.

5 Given the matrix: [5]

$$A = \begin{pmatrix} 3 & 5 \\ 7 & 1 \end{pmatrix}$$

write down the characteristic polynomial and hence determine the eigenvalues of A.

6 Consider the divergence theorem for a volume V given by the cube defined by $0 < x < 1$, $0 < y < 1$ and $0 < z < 1$. For the surface integral on the boundary S of the cube, state the equation for each of the 6 faces (in terms of x, y, z), together with the corresponding outward pointing normal (in terms of $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$). [5]

7 Suppose we have a function $f(x)$ defined as: [5]

$$f(x) = \begin{cases} 5, & 0 \leq x < 2 \\ 0, & 2 \leq x < 5 \\ -1, & 5 \leq x < 10 \end{cases}$$

Expand on the interval $-10 < x < 10$ to make:

a) An even function.
b) An odd function.
c) A function that is neither odd nor even.

8 This is a question on the wave equation. [5]

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}.$$

For a string of length L which is clamped at x = 0 and x = L subject to initial conditions of:

$$y(x,0) = 10\sin\left(\frac{5\pi x}{L}\right) + 15\sin\left(\frac{12\pi x}{L}\right)$$

And

$$\frac{\partial y}{\partial t}(x,0) = 0$$

Solve to find the particular solution.

Section B

Answer TWO questions

9 a) Given a scalar field V, describe the meaning of the gradient $grad\ V$. What does the magnitude and direction of the gradient vector represent? If you are going at an angle θ with respect to the gradient vector, what is the slope of V that you will observe?

[5]

b) In the case of a vector field $\boldsymbol{a}$ representing fluid flow velocity, describe the meaning of the divergence $div\ \boldsymbol{a}$. What does either side of the divergence theorem mean in terms of flow?

[5]

c) Given a vector field:

$$\boldsymbol{a} = (x^2 + y^2)\boldsymbol{i} + 3xy\boldsymbol{j}$$

verify Green's theorem (by evaluating both the line integral and the surface integral) for the triangular region bounded by the lines $y = 0$, $x = 1$, $y = 3x$.

[10]

10 a) The real Fourier series of a function $f(x)$ with a period of 2π can be expressed as:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{n=\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

Where the coefficients a_0, a_n, b_n can expressed as;

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

Where n is an integer number,0,1,2,3... etc

Explain how these generic integral expressions are derived. [7]

b) Suppose that a function $f(x)$ is odd. Explain how you would use this information in the calculation of a Real Fourier series.

[4]

c) For the function defined on the interval $[-\pi, \pi]$

$$f(x) = \{2x\ for\ -\pi < x < \pi\}$$

Calculate the Fourier coefficients and write the Real Fourier series up to n = 5 [9]

11 a) A semi-infinite hotplate is orientated in the xy plane, the y = 0 edge is heated such that its temperature is $T_0 x$ and the $x = 0$ and $x = L$ edges are insulated. The $y = \infty$ edge is kept at a temperature of zero. Make a labeled sketch of this problem. [2]

b) The general solutions of Laplace's equation for this set-up are given by

$$T(x,y) = \begin{Bmatrix}\cos(kx)\\ \sin(kx)\end{Bmatrix}\begin{Bmatrix} e^{ky}\\ e^{-ky}\end{Bmatrix}$$

Derive these general solutions by using the separation of variables.

[6]

c) Define boundary conditions for all 4 edges and hence select appropriate general solutions. Give a condition on k. [4]

d) Hence solve to find the coefficients and specific solution of the hotplate. [8]

END OF PAPER