PRIFYSGOL ABERYSTWYTH UNIVERSITY

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DEPARTMENT OF PHYSICS

SEMESTER 2 EXAMINATIONS, MAY 2026

PH23010 – Principles of Quantum Mechanics

Time allowed – 2 hours

- **Full marks can be attained for complete answers to ALL six questions in Section A and TWO questions from Section B**
- **The numbers in brackets after the questions indicate the allocation of marks for each question.**
- **Only calculators from the Casio FX83, FX85 or FX991 range may be used.**
- **A booklet of mathematical formulae and standard constants will be provided.**
- **Only a 1 side of A4 summary sheet is allowed.**
- **MAE MODD I FYFYRWYR GYFLWYNO ATEBION I'R PAPUR HWN NAILL AI YN Y GYMRAEG NEU'R SAESNEG**
- **STUDENTS MAY SUBMIT ANSWERS TO THIS PAPER IN EITHER WELSH OR ENGLISH**

Once seated, students may complete the front cover of the answer book(s) and the attendance slip ONLY.

DO NOT TURN THE EXAMINATION QUESTION PAPER OVER UNTIL INSTRUCTED TO DO SO.

Section A

Attempt ALL questions in the booklet provided

1 State the formula for the momentum operator $\hat{p}$ in terms of the derivative operator $\frac{\partial}{\partial x}$. State the formula for the $\hat{L}_z$ angular momentum operator in terms of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. [5]

2 State the definition of an *even* function and the definition of an *odd* function. For the following functions, state whether they are even, odd or neither: [5]

a) $\cos(x)$

b) e^x

c) $\sin(x)$

3 State the formula for the energy levels of the *hydrogen atom* in terms of the principal quantum number n and the Rydberg energy E_R. Given a value for the principal quantum number n, what are the allowed values for the l quantum number? Given a value for the l quantum number, what are the allowed values for the m quantum number? [5]

4 Describe the quantum mechanical phenomenon of *tunnelling*. Describe why tunnelling cannot happen in classical mechanics. [5]

5 The time-dependent Schrödinger equation for the one-dimensional free particle is given by: [5]

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}.$$

Deduce the dispersion relation

$$\omega(k) = \frac{\hbar k^2}{2m}$$

for a plane wave solution $\Psi = Ae^{i(kx-\omega t)}$.

6 State the Heisenberg Uncertainty Principle and describe the states that saturate the inequality. [5]

Section B

Answer TWO questions

7 a) A particle is contained in a 1-dimensional box $0 \leq x \leq a$ and has a probability density $\varrho(x)$ which vanishes outside the box. The nth moment of the position is then $\langle q^n \rangle = \int_0^a x^n \, \varrho(x) \, dx$.

Explain how we calculate ϱ from the wavefunctions ψ and give the expression for the average position and the uncertainty in position.

[3]

b) A quantum mechanical particle in the box has wavefunction $\phi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$ inside and vanishes outside. Taking the energy observable inside the box to be given by $\widehat{H} = \frac{\hat{p}^2}{2m}$, where $\hat{p} = -i\hbar\frac{\partial}{\partial x}$ is the momentum, show that we have the eigen-relation

$$\widehat{H}\,\phi_n = E_n\,\phi_n$$

and deduce the value of the energy eigenvalue E_n.

[4]

c) Determine the average position and the uncertainty in position for the particle when it is in state ϕ_n.

You may assume the integrals

$$\frac{2}{a}\int_0^a x \, \sin^2\left(\frac{n\pi x}{a}\right) dx = \frac{a}{2},$$

$$\frac{2}{a}\int_0^a x^2 \, \sin^2\left(\frac{n\pi x}{a}\right) dx = \frac{a^2}{3} - \frac{a^2}{2\pi^2 n^2}.$$

[4]

d) It is known that a particle in the box cannot have an uncertainty Δq greater than the value $a/2$. Determine the minimum uncertainty in the momentum.

[4]

8 a) A quantum mechanical particle has average momentum $\langle p \rangle = 0$ and moves as a free particle (i.e., its energy observable is $\widehat{H} = \frac{\hat{p}^2}{2m}$). Show that the uncertainty in momentum is given by

$$\Delta p = \sqrt{2m\langle E \rangle},$$

where $\langle E \rangle$ is the average energy.

[4]

b) A quantum particle in a box has a wavefunction ψ which has the following expansion in terms of the energy eigenstates:

$$\psi(x) = \sum_{n=1}^{\infty} c_n \, \phi_n(x)$$

with coefficients $c_n = \frac{1}{\sqrt{2^n}}$.

Give the probability p_n that a measurement of the energy in state ψ will yield the nth energy eigenvalue $E_n = \frac{\pi^2 \hbar^2 n^2}{2ma^2}$.

[2]

c) Compute the average energy $\langle E \rangle$ and the uncertainty ΔE for the state ψ.

[You may use the following identities without proof:

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = 6,$$

$$\sum_{n=1}^{\infty} \frac{n^4}{2^n} = 150.]$$

[6]

d) Determine the uncertainty in the momentum for the state ψ.

[3]

9 a) Write down Newton's equation of motion for the classical harmonic oscillator in terms of the mass m and the spring constant k. Express the angular frequency ω in terms of m and k. State the time-independent Schrödinger equation for the quantum harmonic oscillator.

[5]

b) Consider the wavefunction

$$\psi = e^{-\frac{q^2}{2}}$$

Show that

$$\left[q + \frac{d}{dq}\right]\psi = 0$$

and

$$\left[q - \frac{d}{dq}\right]\psi = 2qe^{-\frac{q^2}{2}}$$

Which of the two operators $\left[q + \frac{d}{dq}\right]$ and $\left[q - \frac{d}{dq}\right]$ is the lowering operator?

[5]

c) Explain how the raising and lowering operators can be used to calculate the energy levels and wavefunction solutions of the time-independent Schrödinger equation for the quantum harmonic oscillator.

[5]

END OF PAPER