PRIFYSGOL ABERYSTWYTH UNIVERSITY

EXAMINATION FOR ENTRANCE SCHOLARSHIPS JANUARY 2019

MATHEMATICS

Time allowed: 1 hour 30 minutes

- All answers should be written in the answer books provided, including any diagrams, graphs or sketches. Graph paper is not required.
- Answer **all** questions in Section A and **two** questions from Section B.
- Candidates are permitted to use calculators, provided they comply with A level examining board regulations. They must be made available on request for inspection by invigilators, who are authorised to remove any suspect calculators.
- Statistical tables will be provided.

Section A

1. Simplify the following expressions as far as possible, showing your workings clearly.

(a) $\frac{3}{2-\sqrt{3}} + \frac{6}{\sqrt{12}+3}$ [3 marks]

(b) $\log_3 9000 - 3(\log_3 6 + \log_3 5) + 2$ [6 marks]

(c) $\ln(x^2 + x - 2) - \ln(x + 2) - \ln(x - 1)$, where $x > 1$. [4 marks]

2. Radioactive decay of atoms in a sample can be modelled by $N = N_0 e^{-\lambda t}$ where N_0 is the number of atoms originally in the sample, N is the number of atoms remaining (i.e. not decayed) at time t (seconds), and λ is a positive constant.

(a) Suppose that 72% of the original atoms remain at time $t = 10$. Calculate the value of λ, to four decimal places. [4 marks]

(b) Using the value found in (a), find the time at which 5% of the original atoms have decayed. [3 marks]

3. For each of the following functions, determine the range of values of x for which $f(x)$ is an increasing function.

(a) $f(x) = 2x^2 - 14x + 20$, defined for all real x. [3 marks]

(b) $f(x) = 2x^3 - 3(3x^2 + 20x + 5)$, defined for all real x. [5 marks]

(c) $f(x) = (1 - 3x)\sqrt{x}$, defined for all $x > 0$. [4 marks]

4. Consider the function $f(x) = \frac{2+x^2}{\sqrt{x}}$ for $x > 0$.

(a) Find the indefinite integral of $f(x)$. [4 marks]

(b) Calculate the area enclosed between the x-axis, the curve $y = f(x)$, and lines $x = 1$ and $x = 4$. [4 marks]

5. The points A and B have position vectors $\mathbf{a} = 2\mathbf{i} + 10\mathbf{j}$ and $\mathbf{b} = 6\mathbf{i} + 2\mathbf{j}$.

(a) Write the vector $\frac{1}{4}(\mathbf{a} + 3\mathbf{b}) - 4(\mathbf{i} + \mathbf{j})$ in terms of $\mathbf{i}$ and $\mathbf{j}$. [3 marks]

(b) Find the vector $\mathbf{AB}$, and hence calculate the length of the line AB. [2 marks]

(c) Find the position vector of the point P which divides AB in the ratio $3 : 1$. [2 marks]

6. A circle of radius 1 is centred at point O. Points A and B are placed at distance 2 from O, in such a way that the line passing through A and B is tangent to the circle. Two further circles of radius 1 are centred at A and B, respectively.

(a) Sketch a figure of the three circles and the triangle AOB. [4 marks]

(b) Determine the angles and the area of AOB. [5 marks]

(c) What percentage of the area of AOB is covered by the three circles? [4 marks]

Section B

7. Consider the transformation $y = f(x - a)$ of the graph

$$y = f(x) = 1 - x^2,$$

for varying values of the transformation parameter $0 \leq a < 2$. Let P_1 be the point at which the parabola $y = f(x - a)$ intersects the x-axis on the left side of its symmetry axis, and P_2 the point at which it intersects the line $x = 1$. The line L passes through P_1 and P_2.

(a) Find, in terms of a, the coordinates of P_1 and P_2, and the equation of L. [6 marks]

(b) Show that as $a \to 2$, the line L becomes a tangent to the curve $y = f(x - a)$. [2 marks]

(c) For $a = 0$ and $a = 1$, sketch on the xy-plane the parabola $y = f(x - a)$, and the line L, indicating also the two points P_1 and P_2. [4 marks]

(d) Show that for any $0 \leq a < 2$, the area enclosed between the three lines L, $y = 0$, and $x = 1$ is given by $A = \frac{1}{2}a(2 - a)^2$. [2 marks]

(e) Find the maximum value of A for $0 \leq a < 2$, showing that the value you have found is indeed the maximum. [6 marks]

8. Two objects are moving in straight lines on a horizontal plane with constant velocity vectors

$$\mathbf{v}_1 = 3\mathbf{i} - 2\mathbf{j}, \qquad \mathbf{v}_2 = 4\mathbf{i} + \mathbf{j},$$

respectively (given in units of ms^{-1}). At time $t = 0$ s their position vectors (in units of m) are $\mathbf{r}_1 = \mathbf{j}$ and $\mathbf{r}_2 = -\mathbf{j}$, respectively.

(a) Find (i) the point P at which their paths intersect, and (ii) the times at which they arrive at P. Are they going to meet there? [12 marks]

(b) Find the distance d between the two objects as a function of t. [4 marks]

(c) Find the time at which the distance d is minimal. [2 marks]

(d) Calculate the minimum distance between the two objects. [2 marks]

9. Consider a fair 20-sided die with sides numbered 1 to 20. Let X be the random variable corresponding to the value obtained on one roll of the die. Since the die is fair we have $P(X = x) = 1/20$ for $x = 1, 2, \ldots, 20$.

(a) Find the value k such that $P(X \geq k) = 0.45$. [3 marks]

(b) Let X_1, X_2, ..., X_{10} be the random variables corresponding to 10 independent rolls of the die. Let X_{max} be the maximum obtained on 10 rolls of the die.

(i) Find $P(X_{max} \leq 14)$. (Hint: what has to happen for the maximum to be less than or equal to 14?) [2 marks]

(ii) Find a value m such that $P(X_{max} \leq m) = 2^{-20}$. [5 marks]

(iii) Find $P(X_{max} > 17)$. [3 marks]

(iv) Find $P(X_{max} = 12)$, by expressing $P(X_{max} = 12)$ as a difference of two expressions involving inequalities. [2 marks]

(c) If five 20-sided dice are rolled, let Y denote the number of dice showing a 4 or less. Write down the distribution of Y and find $P(Y \leq 2)$. [5 marks]