1872 PRIFYSGOL ABERYSTWYTH UNIVERSITY

ENTRANCE EXAMINATION
FEBRUARY 2024

FURTHER MATHEMATICS

Time allowed: 1 hour 30 minutes

- All answers (including any diagrams, graphs or sketches) should be written on paper, and scanned into a **single** PDF file. Graph paper is not required.
- Answer **all** questions in Section A and **two** questions from Section B.
- Candidates are permitted to use calculators, provided they comply with A level examining board regulations. They must be made available on request for inspection by invigilators, who are authorised to remove any suspect calculators.

Information and relevant formulas

- Powers of complex numbers: $(r(\cos\theta + i\sin\theta))^n = r^n(\cos(n\theta) + i\sin(n\theta))$.
- For motion in a circle of radius r, the transverse velocity is ωr, and the magnitude of the radial acceleration is $\omega^2 r$, where ω is the angular speed.
- Statistical tables will be provided. Note that the tables refer to the **right-hand** tails of the distributions, that is, probabilities of the form $p = \mathbb{P}(X \geq x)$ where X is a random variable and x an **upper** percentage point of its distribution.
- Formulas related to standard distributions (e.g. for probability, mean, and variance) can be found on the back page of the statistical tables.

Section A

1. Simplify the following expressions as far as possible, showing your workings.

(a) $\dfrac{1-2i}{(1-3i)^2} - \dfrac{1}{25}$; [4 marks]

(b) $\left|3(2\mathbf{i}-\mathbf{j}+3\mathbf{k}) - (2\mathbf{i}-\mathbf{j}-\mathbf{k})\right| - \left|2\mathbf{i}+\mathbf{j}-\mathbf{k}\right|$; [5 marks]

(c) $\left(\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix} - \begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}^{-1}\right)^{-1}$. [5 marks]

2. Solve the equation $2z - \overline{z} = 2\overline{z} + 9 - 2i + z$, for the complex number z. (Here $\overline{z}$ is the complex conjugate of z.) [5 marks]

3. Determine the value of a such that the matrix $\begin{pmatrix}1 & 2\\ 3 & a\end{pmatrix}$ has no inverse. [3 marks]

4. Consider the complex number $z = \sqrt{3} - i$.

(a) Express z in the polar form, that is, as $z = r(\cos\theta + i\sin\theta)$ where $r > 0$ and $-\pi \le \theta \le \pi$. [4 marks]

(b) Express (i) z^2 and (ii) z^7 in the polar form. [5 marks]

(c) Represent z and z^2 as points in an Argand diagram, indicating clearly the relevant angles and radial lines. [3 marks]

(d) Find the smallest positive integer n for which $|z^n| > 10^4$, explaining your reasoning. [3 marks]

5. The complex number z satisfying $|z - 5| = |2\overline{z} + 3i|$ is represented by the point $P(x, y)$ in an Argand diagram. Find the equation of the locus of P in terms of x and y, and interpret it geometrically. [8 marks]

6. Consider the polynomial $f(x) = x^3 + 2x^2 + 3x + c$, where c is a constant.

(a) Determine the value of c such that $f(-1) = 0$. [2 marks]

(b) For the value of c found in (a), find all the roots of the equation $f(x) = 0$, explaining your method for each root. [6 marks]

7. Prove by mathematical induction that

$$\begin{pmatrix}5 & 0\\ 2 & -1\end{pmatrix}^n = \begin{pmatrix}5^n & 0\\ \frac{1}{3}(5^n - (-1)^n) & (-1)^n\end{pmatrix},$$

for all positive integers n. [7 marks]

Section B

8. The points $A(2, 1, 1)$, $B(3, 1, 2)$, and $C(2, 2, 1)$ lie in the plane Π. Line L intersects Π at point C and is perpendicular to Π.

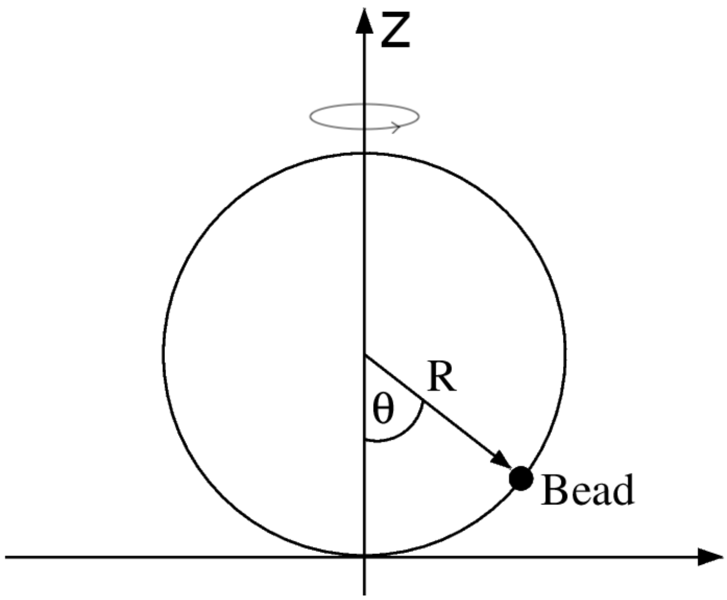
(a) Find (in terms of $\mathbf{i}, \mathbf{j}, \mathbf{k}$) the vectors $\mathbf{AB}$, $\mathbf{BC}$, and $\mathbf{CA}$. [2 marks]

(b) Find the equation of the plane Π in the form $\mathbf{r} \cdot \mathbf{n} = 1$. [7 marks]

(c) Find the angles (in radians to 3dp) and the lengths of the sides of the triangle ABC. [7 marks]

(d) Point D lies on L in such a way that BCD is an isosceles triangle. Find (in radians) the angle $\angle ADC$. [4 marks]

9. A bead of mass m is placed on a frictionless vertical circular track. The track rotates with a constant angular speed around the vertical z-axis; see the figure below. The bead (which moves with the track) then stays in a fixed equilibrium position relative to the track, given by the angle θ.



(a) Draw a free-body diagram for the bead, indicating all the forces acting on it. Also indicate the horizontal and vertical components of the forces when relevant. [4 marks]

(b) Using the free-body diagram, show that $\tan\theta = a_{rad}/g$, where a_{rad} is the radial acceleration towards the z-axis (and g is the gravitational acceleration). [7 marks]

(c) Let T be the time of one full revolution of the track. Express T in terms of θ and R (and g). [7 marks]

(d) From your answer in (c), find the value of T for $\theta = \pi/2$ radians. Explain in a few words what this mathematical result means in the context. [2 marks]

10. (a) Alfred is standing by a bus stop recording the numbers of buses that pass in each 10-minute interval. In 40 ten-minute intervals, his results are as follows:

Number of buses	0	1	2	3	4	5
Frequency	8	7	12	10	2	1

(i) From the given data, calculate the observed mean number of buses passing in a 10-minute interval. [3 marks]

(ii) Alfred wishes to test the goodness of fit of a Poisson distribution with rate parameter given by the answer to part (i). Use this Poisson distribution to calculate (to 2dp) the expected frequencies for observing 0, 1, 2 or 3+ buses in a 10-minute interval. [5 marks]

(iii) Carry out an appropriate goodness of fit test at a 5% significance level (pooling the results for 3, 4, or 5 buses into the 3+ category) to assess whether a Poisson distribution with rate parameter calculated in part (i) is a good fit for the observed data. Make sure to state the null hypothesis, the formula of the χ^2-statistic you use, the degree of freedom, the critical value for the test, as well as your conclusion. [6 marks]

(b) Two independent random variables, X and Y, have expectations $\mathbb{E}(X) = 4$ and $\mathbb{E}(Y) = 7$ and variances $\text{Var}(X) = 3$ and $\text{Var}(Y) = 4$. Calculate the following:

(i) $\mathbb{E}(2X - Y)$; [2 marks]

(ii) $\text{Var}(3X)$; [2 marks]

(iii) $\text{Var}(2X - 3Y)$. [2 marks]