1872 PRIFYSGOL ABERYSTWYTH UNIVERSITY

SCHOLARSHIP EXAMINATIONS
FEBRUARY 2025

FURTHER MATHEMATICS

Time allowed: 1 hour 30 minutes

- All answers (including any diagrams, graphs or sketches) should be written on paper, and scanned into a **single** PDF file. Graph paper is not required.
- Answer **all** questions in Section A and **two** questions from Section B.
- Candidates are permitted to use calculators, provided they comply with A level examining board regulations. They must be made available on request for inspection by invigilators, who are authorised to remove any suspect calculators.

Additional information and relevant formulas

- The polar form of a complex number z is $z = r(\cos\theta + i\sin\theta)$ where $r > 0$ and $-\pi \leq \theta \leq \pi$. Then $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$ for any positive integer n.
- Statistical tables will be provided. Note that the tables refer to the **right-hand** tails of the distributions, that is, probabilities of the form $p = \mathbb{P}(X \geq x)$ where X is a random variable and x an **upper** percentage point of its distribution.
- Formulas related to standard distributions (e.g. for probability, mean, and variance) can be found on the back page of the statistical tables.

Section A

1. Simplify the following expressions as far as possible, showing your workings.

(a) $\dfrac{1}{2-i}+\dfrac{3}{1-2i}-\dfrac{7}{5}i$; [4 marks]

(b) $\left|5\mathbf{i}-4\mathbf{j}+9\mathbf{k}\right|+\left|2(-\mathbf{i}+3\mathbf{j}-\mathbf{k})-(3\mathbf{i}+2\mathbf{j}+7\mathbf{k})\right|$; [5 marks]

(c) $\mathbf{A}+(1-a^2)(\mathbf{I}+\mathbf{A})^{-1}$, if $\mathbf{I}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $\mathbf{A}=\begin{pmatrix}0&a\\a&0\end{pmatrix}$, and $a^2\neq 1$. [6 marks]

2. Solve the equation $(1+3i)\overline{z}-(2+i)z+13=16i$, for the complex number z. (Here $\overline{z}$ is the complex conjugate of z.) [8 marks]

3. Determine all values of a such that the matrix $\begin{pmatrix}2&3a\\3a&a\end{pmatrix}$ has no inverse. [4 marks]

4. Consider the complex number $z=1+i\sqrt{3}$.

(a) Express the four numbers z, $\overline{z}$, $1/z$, and $1/\overline{z}$ in the polar form (as given in the front page). [9 marks]

(b) Represent the four numbers in (a) as points in an Argand diagram, indicating clearly the relevant angles and radial lines. [4 marks]

(c) Express $(2/z)^{40}$ in polar form. [4 marks]

5. The complex number z satisfying $|4i+z|=|\overline{z}-9+8i|$ is represented by the point $P(x,y)$ in an Argand diagram. Find the equation of the locus of P in terms of x and y, and interpret it geometrically. [6 marks]

6. Consider the polynomial $f(x)=x^3+5x^2-4x+c$, where c is a constant.

(a) Determine the value of c such that $f(1+3i)=0$. [4 marks]

(b) For the value of c found in (a), find all the roots of the equation $f(x)=0$, explaining your method for each root. [6 marks]

Section B

7. Let a be any real number, and define the matrix

$$\mathbf{A} = \frac{1}{5}\begin{pmatrix} 4a+1 & 2(a-1) \\ 2(a-1) & a+4 \end{pmatrix}.$$

(a) Solve the equation $\det(\mathbf{A} - x\mathbf{I}) = 0$ for x. (Here $\mathbf{I}$ is the identity matrix.) Note that your solutions may depend on a. [7 marks]

(b) Let

$$\mathbf{P} = \frac{1}{5}\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}.$$

(i) Show that $\mathbf{P}^2 = \mathbf{P}$, including sufficient working to demonstrate that you have verified the claim fully. [2 marks]

(ii) Find a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ such that $\mathbf{P}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$. [3 marks]

(c) Let $\mathbf{P}$ be as in (b). Prove by mathematical induction that

$$\mathbf{A}^n = (a^n - 1)\,\mathbf{P} + \mathbf{I},$$

for all positive integers n. [8 marks]

8. The velocity vector $\mathbf{v}$ (metres/second) at time t seconds of an object of mass 6 kg is

$$\mathbf{v} = 2\sin(5t)\,\mathbf{i} + 2\cos(5t)\,\mathbf{j} - (1+t)^{-2}\,\mathbf{k},$$

where the angle $5t$ is in radians. The object is initially (at time $t = 0$) at position $\mathbf{x} = 2\mathbf{k}$.

(a) Find expressions for the position and acceleration vectors $\mathbf{x}$ (m) and $\mathbf{a}$ (ms^{-2}) of the object at time t seconds. [6 marks]

(b) Find the time at which the object passes through the plane $z = \frac{3}{2}$, and state the coordinates of the crossing point (to 2dp). [4 marks]

(c) Calculate the kinetic energy of the object at time t seconds. [4 marks]

(d) Calculate (to 2dp) the work done by the force acting on the object during the first second (i.e. from $t = 0$ to $t = 1$ second). [3 marks]

(e) Describe (in a few words) the nature of the motion of the object for large values of t, justifying your reasoning. [3 marks]

9. (a) The probability distribution of a discrete random variable, X, is given by

x	1	2	3	4
$\mathbb{P}(X = x)$	0.2	0.3	0.4	0.1

(i) Find $\mathbb{E}(X)$ and $\mathrm{Var}(X)$. [5 marks]

(ii) If $Y = 2X + 3$, find $\mathbb{E}(Y)$ and $\mathrm{Var}(Y)$. [2 marks]

(b) Johann is interested in establishing whether a die is fair. He rolls the die 120 times and obtains the following frequencies of the numbers 1 to 6:

Value	1	2	3	4	5	6
Frequency	18	16	19	25	22	20

Carry out a χ^2 goodness of fit test at the 10% significance level using the formula

$$X^2 = \sum_{i=1}^{6} \frac{(o_i - e_i)^2}{e_i},$$

where o_i is the observed number of times the i^{th} value occurs (for $i = 1, 2, \ldots, 6$) and e_i is the expected number of times the i^{th} value occurs if the die is fair. State your null and alternative hypotheses carefully, identify the distribution of X^2 if the null hypothesis is true and present your conclusion with justification. [8 marks]

(c) A random variable, X, has the distribution $\mathrm{Bin}(10, 0.4)$. Find the following probabilities:

(i) $\mathbb{P}(X = 5)$, [2 marks]

(ii) $\mathbb{P}(3 \leq X < 6)$. [3 marks]