PRIFYSGOL ABERYSTWYTH UNIVERSITY

EXAMINATION FOR ENTRANCE SCHOLARSHIPS JANUARY 2017

FURTHER MATHEMATICS

Time allowed: 1 hour 30 minutes

- All answers should be written in the answer books provided, including any diagrams, graphs or sketches.
- Answer all questions in Section A and two questions from Section B.
- Calculators are permitted, provided they are silent, self-powered, without communication facilities, and incapable of holding text or other material that could be used to give a candidate an unfair advantage. They must be made available on request for inspection by invigilators, who are authorised to remove any suspect calculators.

Section A

1. Integrate each of the following with respect to x:

(a) $\dfrac{x}{x^2}$,

(b) $\dfrac{1}{2x+1}$,

(c) $\dfrac{x+1}{x^2+2}$.

[10 marks]

2. Find the largest set of values x for which the following expressions are defined:

(a) $\sin(5x+1) - e^{3x}$,

(b) $\log(-x^2 - x + 2)$.

[10 marks]

3. (a) Sketch the two loci $|z + 2\sqrt{3}| = 2$ and $|z - 2i| = 2$ in an Argand diagram.

(b) Find a complex number z satisfying $|z + 2\sqrt{3}| = |z - 2i| = 2$. Explain why there is only one such number.

[10 marks]

4. A sequence (a_n) is defined by $a_0 = 1$ and $a_{n+1} = 2a_n + 1$ for all $n = 0, 1, 2 \ldots$. Show by induction that $a_n = \sum_{k=0}^{n} 2^k$. [10 marks]

5. (a) Write down a 2×2-matrix R such that

$$R\begin{pmatrix}2\\0\end{pmatrix} = \begin{pmatrix}\sqrt{3}\\1\end{pmatrix}, \qquad R^2\begin{pmatrix}2\\0\end{pmatrix} = \begin{pmatrix}1\\\sqrt{3}\end{pmatrix}, \qquad R^3\begin{pmatrix}2\\0\end{pmatrix} = \begin{pmatrix}0\\2\end{pmatrix}.$$

[8 marks]

(b) What type of transformation in the plane does the matrix R represent? [4 marks]

(c) Determine the matrix R^{121}. [4 marks]

6. Prove that $2ab \leq a^2 + b^2$ for all real numbers a, b. [4 marks]

Section B

7. A curve has equation $y = \frac{1}{2}ax^2 - \frac{1}{2}x^4 + \frac{1}{6}x^6$, where $a \geq 0$ is an additional parameter.

(a) Show that y has a local maximum point at $x = 0$ when $a = 0$, a local minimum point when $a > 0$, and the *only* local minimum point when $a > 1$. [8 marks]

(b) Show that y has exactly three local minimum points when $0 < a < 1$. Express the local minima in terms of a. [8 marks]

(c) Find the global minimum of y in terms of a for $a \geq 0$. [4 marks]

8. A light inextensible string of length 1.4 m has its ends A and B fixed in a vertical line, with the distance between A and B set at 1.0 m. A particle P of mass 5 kg is attached to the string at a distance 0.8 m from A. P moves in a horizontal circle about the point O (O lies on the line AB), with both parts of the string taut, with an angular speed ω rad/s. The acceleration due to gravity is g m/s^2 and you may ignore air resistance.

(a) Find the tension in each portion of the string in terms of ω and g and hence deduce the minimum value of ω. [12 marks]

(b) The angular speed is increased until the tension in PB is equal to g at which point the string PB breaks. If the particle continues to move with the same angular speed, what is the change in its vertical position? [8 marks]

9. (a) In a bag there are 30 balls, 14 blue, 10 red and 6 green. If 5 balls are drawn from the bag *with replacement*, what is the probability that

(i) 3 of them are blue, [2 marks]

(ii) fewer than 3 of the balls are green? [3 marks]

(b) Suppose that each of 40 balls in a bag is either non-stripy or stripy and one of the three colours, blue, red or green, as shown in the following table:

Colour	Non-stripy	Stripy	Total
Blue	10	9	19
Red	3	7	10
Green	6	5	11
Total	19	21	40

If 2 balls are selected *without replacement*, calculate

(i) the probability that both balls are stripy; [1 mark]

(ii) the probability that both balls are stripy given that the first ball is red; [3 marks]

(iii) the probability that at least one of the balls is blue given that both balls are stripy. [4 marks]

(c) Suppose a coin is tossed 12 times and 10 heads are obtained.

(i) What is the expected number of heads if the coin is fair? [1 mark]

(ii) Assuming the coin is fair, find the probability of observing exactly 10 out of 12 heads. [2 marks]

(iii) If X represents the number of heads, and $E(X)$ represents the expected number of heads if the coin is fair from part (c)i, calculate

$$P(|X - E(X)| \geq K),$$

where $K = 10 - E(X)$. [4 marks]