EXAMINATION FOR ENTRANCE SCHOLARSHIPS JANUARY 2018

MATHEMATICS

Time allowed: 1 hour 30 minutes

- All answers should be written in the answer books provided, including any diagrams, graphs or sketches. Graph paper is not required.
- Answer **all** questions in Section A and **two** questions from Section B.
- Calculators are permitted, provided they are silent, self-powered, without communication facilities, and incapable of holding text or other material that could be used to give a candidate an unfair advantage. They must be made available on request for inspection by invigilators, who are authorised to remove any suspect calculators.
- Statistical tables are not required.

Section A

1. Simplify the following expressions:

(a) $\dfrac{1}{1+\sqrt{3}}\left(\dfrac{1}{1-\frac{1}{\sqrt{3}}}-1\right)$, [3 marks]

(b) $\log_6 9 + \log_6 4$, [2 marks]

(c) $\dfrac{x^2-x-2}{3x-6}+\dfrac{6-x^2-x}{3x+9}$. [5 marks]

2. Differentiate the following expressions with respect to x, simplifying your answer as far as possible:

(a) $x^5+\cos x$, [3 marks]

(b) $\frac{1}{2}\ln(1+x^2)$, [4 marks]

(c) $x(1+\tan^2(5x))\cos^2(5x)$. [3 marks]

3. Integrate the following expressions with respect to x:

(a) $e^{3x}+5x^9$, [3 marks]

(b) $(3x+1)^2+\cos x$, [4 marks]

(c) $\dfrac{9}{3x+1}$. [3 marks]

4. (a) Write down the equation of the circle C, with centre at $(2,1)$ and which intersects the x-axis at the point $P_1=(2,0)$. [3 marks]

(b) Write down the equation of the line L which intersects C at the point P_1, and has gradient -2. [2 marks]

(c) Find the second point P_2 at which the line L intersects C. [5 marks]

(d) Sketch the graphs of C and L on the xy-plane, indicating also P_1 and P_2. [5 marks]

5. (a) Find all solutions x (in radians) of the equation

$$(2\tan x\sin x-1)\cos x = 1$$

that also satisfy $0<x<2\pi$. [8 marks]

(b) Show that the equation $e^{-3x}-\sin(x^3)=0$ has at least three roots in the interval $(0,2)$. (**Note**: you are not required to find the roots.) [7 marks]

Section B

6. Consider the function $f(x) = 1/(1 + x^2)$.

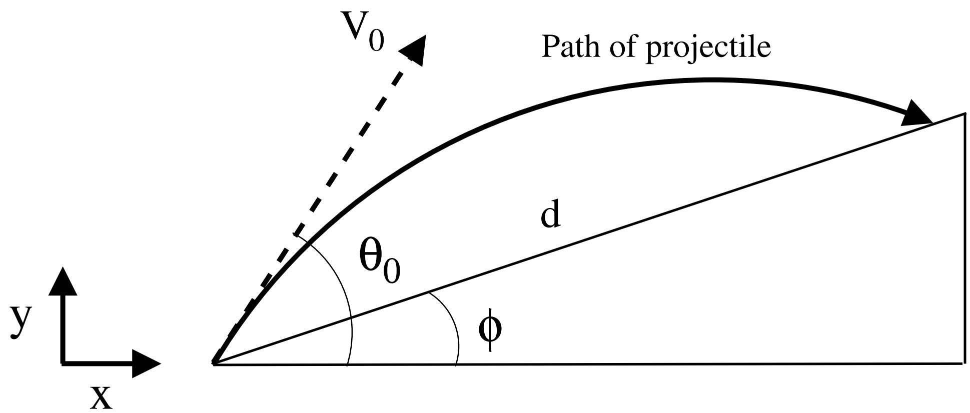
(a) Find the derivatives $f'(x)$ and $f''(x)$, and find a point $x_0 < 0$ such that $f(x_0) = f'(x_0) = f''(x_0)$. [7 marks]

(b) Show that the curves $y = f(x)$ and $y = f'(x)$ have the same tangent line L at x_0, and find the equation of L. [4 marks]

(c) Show that $f(x) \geq f'(x)$ for every real number x. [3 marks]

(d) Calculate the area enclosed by the curves $y = f(x)$ and $y = f'(x)$, and the lines $x = x_0$ and $x = 0$. [6 marks]

7. A projectile is fired up an incline (incline angle ϕ) with an initial speed v_0 at angle θ_0 with respect to the horizontal ($\theta_0 > \phi$) as shown in the figure below.



(a) Show that the trajectory of the projectile is given by the equation

$$y = (\tan\theta_0)x - \frac{g}{2}\frac{x^2}{v_0^2\cos^2\theta_0}$$

where g is the gravitational acceleration.

[3 marks]

(b) Hence show that the projectile travels a distance d up the incline, where

$$d = \frac{2v_0^2\cos\theta_0\sin(\theta_0-\phi)}{g\cos^2\phi}.$$

[7 marks]

(c) For what value of θ_0 is d at its maximum, and what is the maximum value? Simplify your answer for the maximum value as far as possible. [10 marks]

You may require the following identities:

$$\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta),$$
$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta),$$
$$\sin(\alpha)\cos(\beta) = \frac{1}{2}\sin(\alpha+\beta) + \frac{1}{2}\sin(\alpha-\beta).$$

8. Suppose that a continuous random variable, X, has a probability density function given by

$$f(x) = \begin{cases} kx & \text{for } 0 \leq x < 1, \\ 2k - kx & \text{for } 1 \leq x \leq 2, \\ 0 & \text{for } x < 0 \text{ or } x > 2. \end{cases}$$

(a) Find the value of k. [3 marks]

(b) Draw a graph of the function $f(x)$ for $-2 \leq x \leq 4$. [2 marks]

(c) Find $\mathbb{E}(X)$ and $\text{Var}(X)$. [10 marks]

(d) If $Y = 3X + 2$, find $\mathbb{E}(Y)$ and $\text{Var}(Y)$. [2 marks]

(e) Find M such that $P(X \leq M) = 0.4$. [3 marks]