PRIFYSGOL ABERYSTWYTH UNIVERSITY

ENTRANCE SCHOLARSHIP EXAMINATION JANUARY 2018

FURTHER MATHEMATICS

Time allowed: 1 hour 30 minutes

- All answers should be written in the answer books provided, including any diagrams, graphs or sketches. Graph paper is not required.
- Answer **all** questions in Section A and **two** questions from Section B.
- Calculators are permitted, provided they are silent, self-powered, without communication facilities, and incapable of holding text or other material that could be used to give a candidate an unfair advantage. They must be made available on request for inspection by invigilators, who are authorised to remove any suspect calculators.
- Statistical tables will be provided.

Section A

1. Integrate each of the following with respect to x:

(a) $\dfrac{x^2+x}{x}$, [3 marks]

(b) $\dfrac{5}{x-3}$, [2 marks]

(c) $\dfrac{1}{\sqrt{9-x^2}}$. [2 marks]

2. The image of an interval A under a function f is written as $f(A)$ and consists of all $f(x)$ for which x is in A and $f(x)$ is defined. The inverse image of an interval B under a function f is written as $f^{-1}(B)$ and consists of all x for which $f(x)$ is defined and $f(x)$ is in B.

Taking the domain to be the maximal one, for each of the following real functions, find all values in $f((0,1])$ and all values in $f^{-1}((0,1])$:

(a) $f(x) = x^2$, [3 marks]

(b) $f(x) = \ln x$. [3 marks]

3. (a) Sketch the three loci given by $|z-1| = 1$, $|z-e^{i\pi/4}| = 1$ and $|z-1| = |z-e^{i\pi/4}|$ in a single Argand diagram. [5 marks]

(b) Find all complex numbers z satisfying $|z-1| = |z-e^{i\pi/4}| = 1$. [7 marks]

4. A sequence (a_n) is defined by $a_0 = 3$ and $a_{n+1} = \frac{a_n}{a_n+1}$ for all $n = 0, 1, 2 \ldots$. Show by mathematical induction that $a_n = \frac{3}{3n+1}$ for all $n = 0, 1, 2, \ldots$, indicating clearly the relevant steps in the proof. [6 marks]

5. Consider the 3×3 matrices

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

(a) Find AB, BA, and ABA. [3 marks]

(b) Calculate the determinants of A, B and AB. [3 marks]

(c) Calculate A^2, B^2, and hence, or otherwise, find A^{-1} and B^{-1}. [4 marks]

(d) Show that $(AB)^2 = BA$ and $(AB)^3 = I$, where I is the 3×3 identity matrix. Hence find $A^5(AB^3)^{11}(A^3B)^{12}B^{58}$. [5 marks]

6. (a) Factorise $6x^3 + 11x^2 + 6x + 1$ completely into its linear factors. [5 marks]

(b) Resolve

$$f(x) = \frac{26x(x+1)+6}{6x^3 + 11x^2 + 6x + 1}$$

into partial fractions. [5 marks]

(c) Determine the indefinite integral $\int f(x)\,dx$ with $f(x)$ as in (b). [4 marks]

Section B

7. A curve has equation

$$y = \frac{3x^3 - 4x}{2x^2 - 4}.$$

(a) Find the points at which the curve intersects the x-axis. [2 marks]

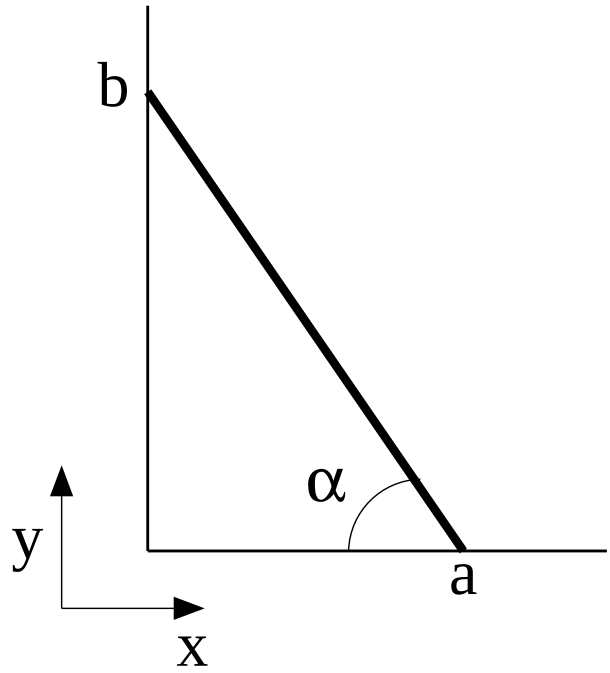
(b) For every stationary point on the curve, find its coordinates and determine whether it is a maximum point or a minimum point. [8 marks]

(c) Determine the equations of the asymptotes of the curve. [5 marks]

(d) Sketch on the xy-plane the curve, the points found in (a) and (b), and the asymptotes found in (c). [5 marks]

8. A ladder is leant against a wall. The ladder makes contact with the floor at the point labelled a and makes contact with the wall at the point labelled b. The angle between the floor and ladder is α as indicated in the diagram below.

The coefficient of the static friction, μ_{sw}, between the ladder and the wall is 0.3 and the coefficient of the static friction, μ_{sf}, between the ladder and the floor is 0.4. The centre of mass of the ladder is in the middle of it.



(a) Draw a free body diagram for the ladder. On the diagram draw and label the following forces acting on the ladder: F_w, the normal force due to the wall; F_f, the normal force due to the floor; F_g, the force of gravity acting through the centre of mass; F_{fw}, the force of friction due to the wall; and F_{ff}, the force of friction due to the floor. [2 marks]

(b) Show that in static equilibrium,

$$F_w - F_{ff} = 0$$

and

$$F_f + F_{fw} - F_g = 0.$$

[2 marks]

(c) By considering the torque about the centre of mass of the ladder, show that

$$F_w \sin\alpha + F_{ff} \sin\alpha + F_{fw} \cos\alpha - F_f \cos\alpha = 0.$$

[4 marks]

(d) Hence find the minimum angle that the ladder can form with the floor for it not to slip down. [12 marks]

9. (a) Suppose in an experiment to evaluate the proportion of animals in a population with a particular genetic trait, 18 animals are sampled, of which 12 have the trait in question. You may assume the population is sufficiently large that we can consider it as sampling **with replacement**.

(i) If θ is the true proportion of animals in the population with the trait, and X is the number of animals with the trait in a sample of size 18, what is the distribution of X? [1 mark]

(ii) Suppose the investigators are interested in establishing whether $\theta > 1/2$. Carry out a hypothesis test for the parameter θ, clearly stating the null and alternative hypotheses, the test statistic, the p-value and the conclusion. (Use the table on page 5 of the statistical tables.) [5 marks]

(iii) In part (ii) what is the critical region (i.e. the range of values of X) for which the investigators would reject the null hypothesis in favour of the alternative hypothesis at the 1% level? Justify your answer. (Again you will need the table on page 5 of the statistical tables.) [3 marks]

(b) Let $\theta > 0$ be a given constant. Suppose that the random variables X and Y are **independently distributed** uniform (rectangular) random variables, where

$$X \sim U[0, \theta]$$

and

$$Y \sim U[0, \theta/2].$$

Let a and b be two real numbers and let $Z = aX + bY$. Calculate the expectation and variance of Z, showing your working clearly. [11 marks]